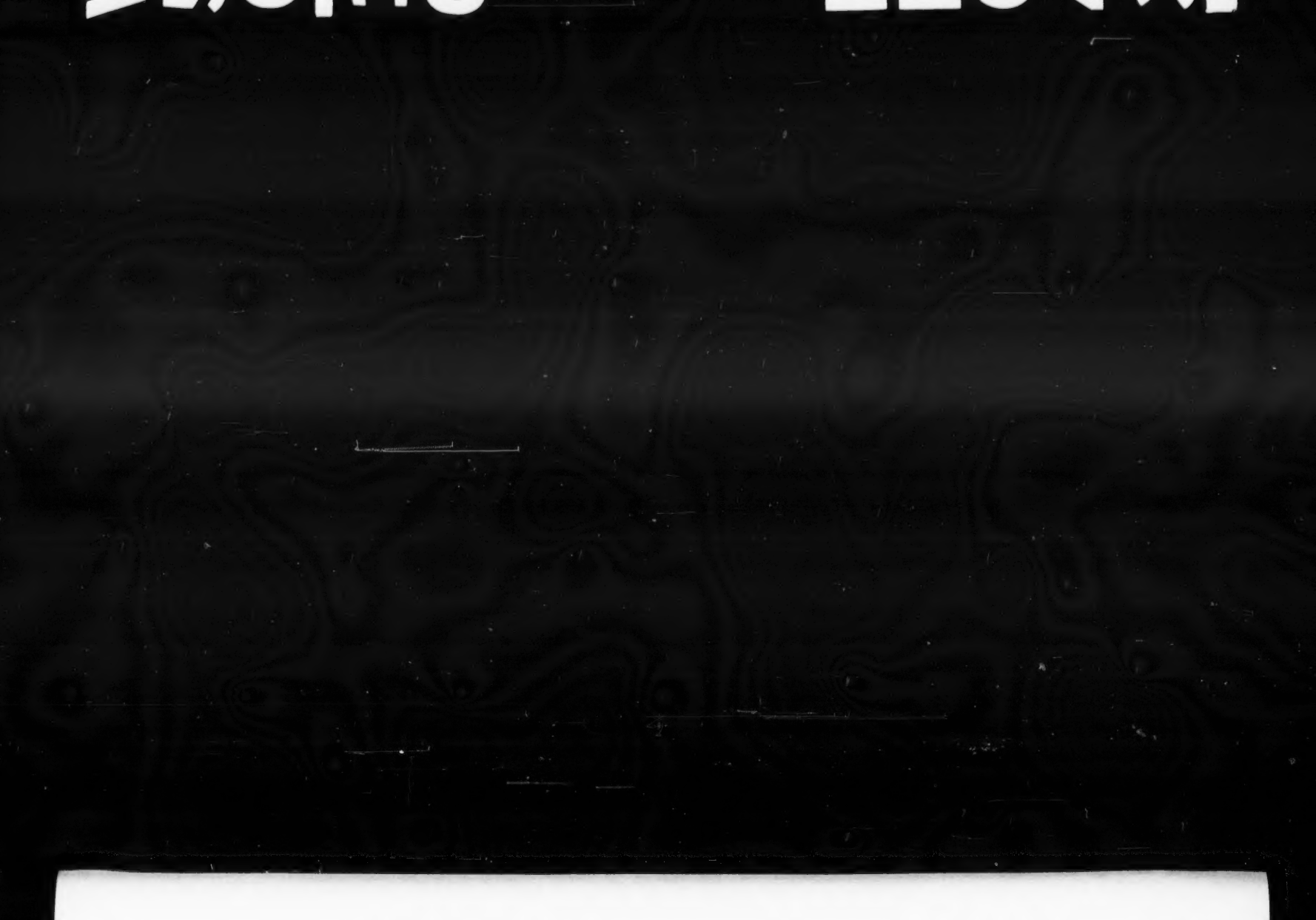

WHAT IS MEASURE?

S. M. ULAM, University of Wisconsin

The concept of measure includes the notions generalizing the old ideas of length, area and volume of figures; all of which are among the oldest in mathematics and, in fact, as basic as the idea of number itself.

1. The elementary approach. The first mathematical approach to the idea of length of a curve, area of a surface, or volume of a solid consisted in assuming them as "evident" or given for some elementary figures, for instance line segments, rectangles, parallelepipeds or prisms. One calculated these numbers for polygons or polyhedra by decomposing them into elementary figures; and for general curved figures, passages to the limit were used. Archimedes followed this procedure for the computation of areas and volumes of a few of the simplest geometrical objects. The question of the consistency of this method was, of course, raised much later. Not until the development of the infinitesimal calculus, which gave the necessary tools for working systematically with the limit processes, was the class of figures whose areas or volumes could be defined and calculated significantly enlarged.

Cauchy, Riemann and Jordan gave rigorous definitions and statements of properties of the definite integral, and thus gave solid foundations for a treatment of the areas and volumes of the objects that formed the domain of study of most of the 19th century mathematics—sets defined by inequalities satisfied by continuous functions on the n-dimensional Euclidean space. [Let us point out here that in general the problem of the integral and the problem of measure are intimately related. A general notion of integral permits one to define a general measure, and vice-versa.] In the second half of the 19th century, in many parts of mathematics, it became necessary to investigate more general sets of points in Euclidean space. In the study of trigonometric series, in function theory, and especially in the investigations of Poincaré in the theory of probability and the general theory of dynamical systems, there appeared sets of points defined by discontinuous functions (obtained through passages to the limit effected on continuous functions).

Cantor's creation of set theory, where the notion of a geometrical figure was generalized into that of an arbitrary subset of points of a given space, introduced also a need for an axiomatic investigation of the problem of measure of sets and, at the same time, made possible a logical analysis of the notion of measure in general.

2. Lebesgue's measure. Borel, and above all Lebesgue [0], applied the ideas of set theory to the problem of measure. Lebesgue's procedure for introducing a measure for sets situated in the Euclidean space was essentially this:

There is given a collection (class) of sets situated in the Euclidean space. This class contains the elementary figures and is large enough to include all sets that can be obtained by the processes usually employed in analysis; in particular the complement of a set that is in the class also belongs to the class, and

the union of any denumerable number of sets in the class yields sets belonging to this class (the denumerable union of sets corresponds to the process of summing infinite series). One has to attach to every set in the class a non-negative, real number, called its measure, so that the following postulates will be satisfied:

I. All sets of a specific subclass should have measure, for example, all sets consisting of a single point.

II. The measure of a set should coincide with its ordinary value in the case when the set is an elementary figure.

III. The postulate of additivity: two forms, a weaker and a stronger form are possible. This requires that the measure of a finite (or in a stronger form, denumerably infinite) sum of mutually disjoint sets should be equal to the numerical sum of the measures of the individual sets.

IV. The invariance or congruence postulate: This requires that sets congruent in the sense of elementary geometry, should have equal measure.

The smallest class of sets for which it is meaningful to discuss this general problem of measure, consists of the so-called Borel sets. These are all sets that one can obtain from intervals (or parallelepipeds in the n-dimensional case) by the two operations of taking a complement of a set and taking infinitely denumerable sums of sets—repeated any number of times. Lebesgue recognized that one can enlarge this class by considering all sets that differ from a Borel set by a subset of any Borel set of measure. He succeeded in giving a constructive definition of a measure of this kind. His measure meets many but, as we shall see, not all the needs of analysis.

We give these well known historical facts because, even in this abbreviated and schematic form, they throw light on the developments that followed Lebesgue.

3. The abstract point of view. If we examine the problem critically, the folowing questions arise at once:

How large can the class of sets be for which a measure, in the sense given above, can be defined? Lebesgue's class is closed under the operations of denumerable addition and intersection of sets, and so meets many but not all of the situations arising in analysis. Using simply the operation of projection of a Borel set from the n-dimensional space into the $n-1$ dimensional space (for example from the plane into the straight line) one can define sets that do not belong to Borel's class. By operating again with complements of such sets and projections, one obtains a wide class of sets, the so-called projective sets. Whether these belong to Lebesgue's class is still an open question. Such sets arise very naturally, especially in function spaces. It seems very natural to include the operation of projection of sets (or what would be the same, taking the images through continuous transformations) in the consideration of the class of measurable sets. Lebesgue's class certainly does not contain all sets. This was shown first by Vitali [1].

The n-dimensional Euclidean space still remains the most important space of mathematics, but it is only the most important special case among the many spaces studied in geometry and analysis. It may be variously looked upon as an example of a general topological space, of a group manifold, of a Riemannian space, or of a finite dimensional vector space. Is a theory of measure for subsets of these spaces possible? The great development of the theory of probability creates the need of a theory of measure for sets in general "phase spaces" some of which are infinitely dimensional function spaces.

In analogy with the property of invariance of Lebesgue's measure for congruent sets, we might have to examine, in spaces more general than the Euclidean, the various meanings of congruence, and the corresponding properties of a measure.

Let us study all these points in greater detail.

4. The problem of the class of measurable sets. Let us make clear from the beginning that this discussion is necessary only because it is impossible to have measure defined for all subsets of a set [2]. To make things precise, we shall examine the situation where all sets for which a measure has to be defined, are subsets of the interval $0 \leqq x \leqq 1$. The class of sets should be such that the sums of denumerably many sets belonging to it also belong to the class, and the complement of a set in the class is also in the class. We seek a real valued function $m(A)$, the measure of the set, having the following properties:

I. $m(E)=1$, where E denotes the entire interval $[0-1]$.

II. $m(p)=0$, when p denotes a set composed of any single point p.

III. The additivity property, that is, the measure of the sum of two disjoint sets, is equal to the numerical sum of the measures: $m(A+B)=m(A)+m(B)$ whenever $A \cdot B=0$.

III′. The additivity for a denumerably infinite number of disjoint sets: $m(\sum_{i=1}^{\infty} A_i)=\sum_{i=1}^{\infty} m(A_i)$ if $A_i \cdot A_j=0$ for $i \neq j$.

It is impossible to have a measure function defined for all sets on the interval with the properties I, II, III′ [2]. But if one requires only I, II, III, one can define a measure with these properties for *all* subsets, and as a matter of fact, one can have an additional property of the measure, namely, that any two sets congruent in the sense of elementary geometry, will have equal measure [3].

Quite generally it is possible to have a finitely additive measure function, in any additive class of sets, the measure assuming only the two values 0 and 1, and being equal to 0 for all sets of a *prescribed* additive subclass of the given class [4].

The fact that it is not possible to have an infinitely additive measure for all sets on the interval, makes it necessary, *even in the case where one does not require invariance or congruence properties* for the measure to consider *subclasses* of measurable sets; we see that the problem is one of algebra and set theory at this stage. The problem of determining for what additive classes of sets there will exist a real valued infinitely additive set function is not yet solved. An answer

can be given in the case when the given class is generated by denumerably many sets (generated by the operations of taking the complement of a set and adding denumerably many sets in the class) [5].

5. Invariant measures. The problem of measure becomes more interesting when one requires that sets that are congruent or equivalent should have the same measure.

We shall now explain the meaning of congruence or equivalence by means of a few examples.

If the space E is Euclidean or more generally metric, the congruence of two sets A and B means that one can map A into B by a one-to-one transformation, so as to preserve distances between pairs of points. (The transformation needs to be defined only on A, and not necessarily on the whole space E.) It is natural to require that any two congruent sets should have equal measure.

Another case: If the space E is Euclidean, we have a group of point transformations of E into itself, for example, the rigid motions. In general, when E is a manifold on which a given group $\mathfrak{G}$ of point transformations is specified, one might call two sets A and B equivalent if there is a transformation T in the group $\mathfrak{G}$ such that $T(A)=B$. In any space E that is also a group manifold a group of transformations is obtained in a natural way, for example, by multiplying all elements x of the space by a fixed element a on the left: $T(x)\equiv a\cdot x$.

Finally, we could conceive in the most general way of equivalence among subsets of a given space E as a given (but otherwise arbitrary) reflexive and transitive relation between sets, thus dividing the sets into classes of mutually "equivalent" ones.

In all these cases one can ask for a measure function which, in addition to the additivity properties, would assign equal values to equivalent sets.

Let us indicate briefly the present state of knowledge concerning the existence of measures with specified invariance properties. Lebesgue's measure is defined only for certain sets, but it has the properties I, II, III′, IV [6]. Banach has constructed a measure function for all sets on the real line, or all sets in the Euclidean plane with properties I, II, III, IV. The fact that a measure of this sort is not possible for all subsets of the Euclidean space of *three dimensions* was first proved by Hausdorff [7]. A later proof was given by Banach and Tarski [8] in their famous paradoxical decomposition of two solid spheres, of different radii. Each sphere can be decomposed into the same finite number of mutually disjoint sets, the sets forming the bigger sphere being respectively congruent to those used in the decomposition of the smaller sphere!

In a very interesting paper [9] Von Neumann has discussed the problem of measure for all subsets of any given group (thus generalizing the problem from the Euclidean vector groups to an arbitrary group) and characterized those groups for which a measure function satisfying I, II, III, IV can be defined for all subsets of the group. Let us note that IV refers to congruence as defined above for a general group.

Since in general we cannot define a measure function for all subsets, we have to consider special classes of subsets. In an important paper [10] A. Haar has established the existence of a denumerably additive and invariant measure function for subsets of any locally compact topological group. The class of sets for which he succeeded in defining his measure includes all Borel sets (we recall that these are sets obtainable from open sets by the operations of complementation and denumerable summation).

The problem of the existence of a measure function was treated for a general notion of equivalence of sets by Tarski [11]. He obtained necessary and sufficient conditions under which, if *one requires finite additivity only*, a measure function satisfying the congruence postulate is possible. But the solution of this general problem for denumerably infinite additivity is yet to be obtained. Special notions of equivalence for sets, and the existence of a measure having equal values for equivalent sets, have been considered by several others [12].

We cannot here enter into the discussion of properties and applications of measure functions. Let us remark, however, that in many cases the properties which we have postulated for measure determine it uniquely. Lebesgue himself showed that a measure function for sets in the Euclidean space satisfying I, II, III, IV, must necessarily coincide with the one he has constructively defined. This also holds for measures in Haar's groups [13].

Limitations of space forbid the discussion of applications of measure theory to the general theory of probability where one deals with some of the most abstract aspects of the theory and constructs measures in composite spaces, using given measures in given spaces [14], or to ergodic theory [15]. For connections between measures and topologies in algebraical structures the reader is referred to the book of A. Weil [10]. Likewise, we cannot discuss here the general point of view under which one would define measures, not necessarily for sets that are subsets of a given space, but rather for elements of a general Boolean algebra [16]. Also one could study measures with other than real values; p-adic values for example seem indicated in certain general situations arising in topology.

6. The general problem. To summarize, we note that measure is a set function associating a non-negative, real number with every set in a certain class of sets, all forming subsets of a given space. Characteristic properties of measure are its additivity for disjoint sets (finite additivity or, if one can obtain it, denumerably infinite additivity). The other characteristic property is equality of measure for sets that are congruent or otherwise considered equivalent, the notion of congruence or equivalence being provided by the geometry or algebra of the given situation. The problem of existence of such set functions has been solved in many special cases, including some most important ones in Euclidean space. However the general problem, that is, to determine whether or not, in a given class of sets with a given equivalence relation, a measure is possible, seems very difficult.

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AN ELEMENTARY PROOF OF THE BUDAN-FOURIER THEOREM

N. B. CONKWRIGHT, University of Iowa

1. Introduction. It is possible that the treatment of the Budan-Fourier theorem outlined here may be of interest to those who teach introductory courses in the theory of equations.

Let

$$f(x) = 0 \tag{1}$$

be a polynomial equation of degree n with real coefficients. Let V_c denote the number of variations of sign in the sequence

$$f(x), f'(x), f''(x), \cdots, f^{(n)}(x), \tag{2}$$

when $x = c$, where c is any real number. The customary statement of the theorem is as follows:

Let a and b be any two real numbers, $a < b$. Then the number of roots of (1) *in the interval $a < x \leqq b$ (an m-fold root being counted m times) is $V_a - V_b - 2k$, where k is a positive integer or zero.*

In elementary text-books the theorem is sometimes stated in a simplified form obtained by imposing the condition that neither $f(a)$ nor $f(b)$ is zero.

It is a familiar fact that Descartes' rule of signs is an immediate consequence of the Budan-Fourier theorem. It is equally well known that Descartes' rule can be established independently by an argument of an elementary nature. The following proof of the Budan-Fourier theorem will be based upon Descartes' rule. The method used is similar to an argument which the writer has previously employed to derive Descartes' rule.*

2. Proof of the theorem. In this discussion the interval $a < x \leqq b$ will be denoted by I. The symbols h_1, h_2, s, and t will be used to denote quantities each of which is either a positive integer or zero. The number of roots of (1) in the interval I will be denoted by p.

It is easily seen that p is equal to the number of positive (non-zero) roots of

$$f(x + a) = f(a) + f'(a)x + \tfrac{1}{2}f''(a)x^2 + \cdots + \frac{1}{\lfloor n}f^{(n)}(a)x^n = 0 \tag{3}$$

minus the number of positive (non-zero) roots of

$$f(x + b) = f(b) + f'(b)x + \tfrac{1}{2}f''(b)x^2 + \cdots + \frac{1}{\lfloor n}f^{(n)}(b)x^n = 0. \tag{4}$$

This follows from the fact that equations (3) and (4) are obtained by diminishing the roots of (1) by a and b respectively.

* See the author's Introduction to the Theory of Equations, pp. 48–49.

By virtue of Descartes' rule it is known that (3) has V_a-2h_1 positive roots, and that (4) has V_b-2h_2 positive roots. Therefore

$$p = (V_a - 2h_1) - (V_b - 2h_2) = V_a - V_b - 2k, \tag{5}$$

where $k=h_1-h_2$. To establish the Budan-Fourier theorem we need only prove that $k\geqq 0$.

The proof that $k\geqq 0$ will be effected by mathematical induction. It can easily be verified that $k\geqq 0$ if $f(x)$ is of the first degree. Hence it will be sufficient to show that this inequality holds for every equation of degree n if the corresponding inequality holds for every equation of degree $n-1$. We shall therefore base the argument to follow upon the assumption that the specified inequality is valid for every equation of degree $n-1$.

Now let q denote the number of roots of the equation $f'(x)=0$ in the interval I, and let V_c' denote the number of variations of sign in the sequence

$$f'(x), f''(x), \cdots, f^{(n)}(x),$$

when $x=c$, where c is any real number. By an argument similar to that which led to (5) it can readily be shown that

$$q = V_a' - V_b' - 2k'$$

where k' is an integer or zero. And by virtue of our assumption it may be affirmed that $k'\geqq 0$.

It follows from Rolle's theorem that $q\geqq p-1$, say $q=p-1+s$. We note also that

$$V_a = V_a' \quad \text{or} \quad V_a = V_a' + 1,$$

and

$$V_b = V_b' \quad \text{or} \quad V_b = V_b' + 1.$$

An examination of the various possibilities will reveal that

$$V_a - V_b \geqq V_a' - V_b' - 1. \tag{6}$$

Consider first the case in which the inequality sign holds in (6). Then

$$V_a - V_b \geqq V_a' - V_b' = q + 2k' = p - 1 + s + 2k'. \tag{7}$$

Now if $s+2k'\neq 0$, we have $V_a-V_b\geqq p$. It follows from (5) that $k\geqq 0$, as was to be proved.

If $s+2k'=0$, we have $V_a-V_b\geqq p-1$. But it follows from (5) that V_a-V_b-p is *even*, and hence $V_a-V_b\neq p-1$. Consequently $V_a-V_b>p-1$, and therefore $V_a-V_b\geqq p$. Again it follows from (5) that $k\geqq 0$.

Consider next the case in which the two members of (6) are equal. This is possible only if $V_a=V_a'$, and

$$V_b = V_b' + 1. \tag{8}$$

It will be shown in the next paragraph that, if (8) holds, then $q \geqq p$. Accepting this fact for the moment, and writing $q = p + t$, we have

$$V_a - V_b = V_a' - V_b' - 1 = q + 2k' - 1 = p + t + 2k' - 1. \tag{9}$$

The first and last members of (9) are essentially the same as in (7). And by an argument similar to that developed in connection with (7) it can be shown that $k \geqq 0$ in the case under consideration.

To complete the proof of the Budan-Fourier theorem it will now suffice to show that $q \geqq p$ if (8) holds. In this case $f(b) \neq 0$, and if $f'(b) \neq 0$, the quantities $f(b)$ and $f'(b)$ must have unlike signs. Now let r be the greatest root of (1) in the interval I. Then $f'(x)$ must vanish at least once in the interval $r < x \leqq b$, for otherwise $f(b)$ and $f'(b)$ would have the same sign. (This is obvious from graphical considerations, and can also be proved by a simple argument based upon the mean value theorem.) Hence the equation $f'(x) = 0$ has in the interval I the $p - 1$ roots vouched for by Rolle's theorem, and at least one additional root in the interval $r < x \leqq b$. Therefore $q \geqq p$.

A PLANE REPRESENTATION OF VECTORS AND TENSORS*

F. P. BEER, Williams College

1. The methods of plane representation of space vectors. Two methods are used for the plane representation of a free vector $\mathbf{P}$ of components X, Y, Z. The first method, introduced by B. Mayor† and developed by R. von Mises‡, associates with $\mathbf{P}$ a line-bound†† vector $\mathbf{P}'$ of the plane XOY, whose x- and y-components are equal to X and Y respectively, and whose moment with respect to OZ is equal to cZ, c being a constant. The second method, used by W. Prager§, associates with $\mathbf{P}$ a line-bound vector $\mathbf{P}''$ normal to the plane XOY, and such that the z-component of $\mathbf{P}''$ is equal to Z, and its moments with respect to OX and OY are equal to cX and cY respectively. The vectors $\mathbf{P}'$ and $\mathbf{P}''$ both characterize $\mathbf{P}$ and are called the first and the second images of $\mathbf{P}$.

In both cases the system formed by the images of given vectors of the space is equivalent to the image of the sum of the given vectors. Furthermore, using the two methods simultaneously, W. Prager** has established formulas expressing

* This work was carried out at Brown University, Providence, R. I., with suggestions from Professor W. Prager, during the Summer Session of Advanced Instruction and Research in Applied Mechanics, 1942.

† B. Mayor, Introduction à la statique graphique des systèmes de l'espace, Lausanne, 1910

‡ R. von Mises, Graphische Statik räumlicher Kraftesysteme, Zeitschrift für Mathematik und Physik, vol. 64, p. 209, 1916.

†† I.e. one whose line of action is fixed.

§ W. Prager, Beitrag zur Kinematik des Raumfachwerks, Zeitschrift für angewandte Mathematik und Mechanik, vol. 6, p. 341, 1926.

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the scalar and vector products of two vectors by means of their images. In the following we need only the formula for the scalar product of two vectors $\mathbf{P}_1$ and $\mathbf{P}_2$ represented by their first and second images respectively. Denoting by $\mathbf{r}_1'$ the vector joining O to a point of the line of action of $\mathbf{P}_1'$ and by $\mathbf{r}_2''$ the position vector corresponding to the trace of $\mathbf{P}_2''$ on XOY, we have

$$\mathbf{P}_1 \cdot \mathbf{P}_2 = \frac{1}{c} (\mathbf{P}_2'' \times \mathbf{P}_1') \cdot (\mathbf{r}_2'' - \mathbf{r}_1'). \tag{1}$$

In order to obtain a simpler formulation of our results, we shall slightly modify these methods of representation by rotating the image vectors by $-\pi/2$ around the z-axis. This change of orientation does not alter the law of summation, and formula (1) also remains valid. The first image $\mathbf{P}'$ of the vector $\mathbf{P}(X, Y, Z)$ is then characterized by the two components $X' = Y$, $Y' = -X$ and a line of action p'. The equation of p' can be derived from the condition imposed on the moment of $\mathbf{P}'$ with respect to OZ; it has the form

$$xX + yY + cZ = 0, \tag{2}$$

where x and y are the running coordinates. In a similar way, the second image $\mathbf{P}''$, of modulus Z and normal to XOY, is found to pierce that plane at a point p'' of coordinates

$$x'' = c\frac{X}{Z}, \qquad y'' = c\frac{Y}{Z}. \tag{3}$$

As the line of action p' of the first image and the trace p'' of the second image both characterize the direction of $\mathbf{P}$, a relationship must exist between these two elements. Eliminating X, Y, Z between (2) and (3), we find that p' is the antipolar of p'' with respect to the circle defined by

$$x^2 + y^2 - c^2 = 0.$$

The relation between the two images will be completely determined if we recall that their moduli P' and P'' satisfy the equations

$$P' \cdot d' = cZ \quad \text{and} \quad P'' = Z,$$

where d' is the distance of p' from the origin. It should be noticed that the modulus of an image is to be considered as an algebraic quantity, the sign of which depends upon the sense of the image. Denoting by d'' the distance Op'', we have

$$d' \cdot d'' = c^2,$$

and consequently

$$P'' = \frac{d'}{c} P' \quad \text{and} \quad P' = \frac{d''}{c} P''. \tag{4}$$

These equations express the modulus of one image as a function the of elements of the other. The formulae (4) show that the signs of P' and P'' are alike. This means that $\mathbf{P}''$ and the moment vector of $\mathbf{P}'$ with respect to O have the same sense.

2. Plane representation of symmetric tensors. Consider a symmetric tensor*

$$\theta = \begin{Bmatrix} I_x & -D_z & -D_y \\ -D_z & I_y & -D_x \\ -D_y & -D_x & I_z \end{Bmatrix}. \tag{5}$$

The tensor θ serves to define the transformation of a vector $\omega(\omega_x, \omega_y, \omega_z)$ into a vector $\mathbf{S}=\theta\,\omega$, whose components are

$$\begin{aligned} S_x &= I_x\omega_x - D_z\omega_y - D_y\omega_z, \\ S_y &= -D_z\omega_x + I_y\omega_y - D_x\omega_z, \\ S_z &= -D_y\omega_x - D_x\omega_y + I_z\omega_z. \end{aligned} \tag{6}$$

Associate with $\mathbf{S}$ its first image $\mathbf{S}'$ in the plane XOY, and with ω its second image ω''. The equation of the line of action s' of $\mathbf{S}'$ is obtained by substituting S_x, S_y, S_z for X, Y, Z in (2). Similarly, the coordinates x'', y'' of the trace Ω'' of ω'' on XOY are found by substituting ω_x, ω_y, ω_z for X, Y, Z in (3). Using (6) we obtain the following relation between x'', y'' and the running coordinates x, y of s':

$$I_x x x'' + I_y y y'' - D_z(xy'' + yx'') - D_y c(x + x'') - D_x c(y + y'') + I_z c^2 = 0.$$

Hence s' and Ω'' are polar and pole with respect to the conic q of equation

$$I_x x^2 + I_y y^2 - 2D_z xy - 2D_y cx - 2D_x cy + I_z c^2 = 0. \tag{7}$$

The coefficients of this conic are readily determined from the components of θ, and, as s' and Ω'' characterize the directions of $\mathbf{S}$ and ω respectively, we see that q determines the tensor θ, as far as the directions are concerned.

Compare q with the conjugate quadrics Q and Q' used by Cauchy in his representation of a symmetric tensor. The equation of these quadrics is

$$I_x X^2 + I_y Y^2 + I_z Z^2 - 2D_x YZ - 2D_y ZX - 2D_z XY = \pm h^2, \tag{8}$$

where h is a constant. The equation of the cone K asymptotic to Q and Q' is obtained by setting $h=0$ and we see that q can be considered as the intersection of K with the plane $Z=c$. If Q and Q' are unparted and biparted hyperboloids respectively, K is real, and q is a real conic (ellipse, hyperbola or parabola). If Q and Q' are real and imaginary ellipsoids respectively, as in the case of the tensor of inertia, K is imaginary, and q is an imaginary ellipse. In order to avoid using imaginary curves, we shall then use instead of q its conjugate $\bar{q}$.

* The choice of notations is due to the fact that later θ will denote the tensor of inertia of a rigid body.

The coordinates x_0, y_0 of the center C of q and $\bar{q}$ are obtained by solving

$$(9) \qquad I_x x_0 - D_z y_0 - D_y c = 0, \qquad - D_z x_0 + I_y y_0 - D_x c = 0.$$

Introduce the new coordinates $\xi = x - x_0$, $\eta = y - y_0$. Denoting by Δ the determinant of the matrix (5) and by δ the minor belonging to I_z, the equations of q and $\bar{q}$ may be written respectively as

$$(10) \qquad I_x\xi^2 - 2D_z\xi\eta + I_y\eta^2 + c^2\frac{\Delta}{\delta} = 0, \quad I_x\xi^2 - 2D_z\xi\eta + I_y\eta^2 - c^2\frac{\Delta}{\delta} = 0.$$

One of these conics at least is real, and we can determine its orientation and axes from the corresponding equation (10). A real conic is thus associated with θ, and can be easily constructed.

We know already how to draw the line of action s' when ω'' and the conic q or $\bar{q}$ have been constructed. We now shall show how the modulus and sense of $\mathbf{S}'$ can be determined from the same elements. The moment of $\mathbf{S}'$ with respect to the center C of the conic, is obtained as the sum of the moment cS_z with respect to O and the moment, with respect to C, of a vector equipollent to $\mathbf{S}'$ and attached to O. Hence

$$g \cdot S' = x_0 S_x + y_0 S_y + cS_z,$$

where S' (whose sign characterizes the sense of the moment of $\mathbf{S}'$ with respect to O) is the modulus we want to determine, and g the algebraic distance of s' from C, considered as positive if O and C are on the same side of s'. Using the relations (6) we find

$$gS' = \omega_x(I_x x_0 - D_z y_0 - D_y c) + \omega_y(- D_z x_0 + I_y y_0 - D_x c) + \omega_z(- D_y x_0 - D_x y_0 + I_z c).$$

Equations (9) show that the two first parentheses are equal to zero. The third one is equal to $f(x_0, y_0)/c$, where $f(x, y)$ denotes the first member of the equation (7) obtained for q. Writing $f(x, y)$ as in the first of the equations (10) we obtain

$$\frac{1}{c} f(x_0, y_0) = \frac{c\Delta}{\delta}.$$

Finally, recalling that ω_z is equal to the modulus ω'' of $\boldsymbol{\omega}''$, we find

$$(11) \qquad S' = \frac{c\Delta}{\delta}\frac{\omega''}{g}.$$

The first image of $\mathbf{S}$ is thus entirely determined from the second image of $\boldsymbol{\omega}$. The senses of $\boldsymbol{\omega}''$ and of the moment of $\mathbf{S}'$ in C (of modulus gS') are alike or different according as the quotient of Δ and δ is positive or negative. In the case of the tensor of inertia these two determinants are positive, and $\boldsymbol{\omega}''$ and the moment of $\mathbf{S}'$ with respect to C have the same sense.

3. Scalar product of S and ω. Moment of inertia. We shall now form the scalar product of the vectors S and ω by means of their images S′ and ω″. Drawing the perpendiculars $\Omega''L$ and CL' on s' (see figure), we denote by **u** the vector running from L to Ω'' and by **v** that from L' to C. Choosing L as the extremity of the vector $\mathbf{r}_1'$, formula (1) yields

$$\mathbf{S}\cdot\boldsymbol{\omega} = \frac{1}{c}(\boldsymbol{\omega}'' \times \mathbf{S}')\cdot\mathbf{u}. \tag{12}$$

As ω″ and S′ are perpendicular to each other the modulus of their vector product equals $\omega''\cdot S'$ or, according to (11)

$$\frac{c\Delta}{\delta}\frac{\omega''^2}{g}.$$

The direction of the vector product is that of **v**. The vector product has the sense of **v** if the quotient of Δ and δ is positive. Noting that both $|g|$ and $|\mathbf{v}|$ are equal to $L'C$, we get

$$\boldsymbol{\omega}'' \times \mathbf{S}' = \frac{c\Delta}{\delta}\omega''^2\frac{1}{(L'C)^2}\mathbf{v}.$$

The vectors **u** and **v** being parallel, their scalar product equals $(L\Omega'')(L'C)$, and formula (12) takes the form

$$\mathbf{S}\cdot\boldsymbol{\omega} = \frac{\Delta}{\delta}\omega''^2\frac{L\Omega''}{L'C}. \tag{13}$$

Draw $\Omega''C$, and denote its intersection with s' by N. Then

$$\mathbf{S}\cdot\boldsymbol{\omega} = \frac{\Delta}{\delta}\omega''^2\frac{N\Omega''}{NC}. \tag{14}$$

The two last formulae are valid in the general case of a symmetric tensor. Consider now the particular case of the tensor of inertia. The conic q is then an imaginary ellipse, and s' must be considered as the antipolar of Ω'' with respect to the real ellipse $\bar{q}$. As the center C of $\bar{q}$ is located between N and Ω'',

$$\frac{N\Omega''}{NC} = \frac{NC + C\Omega''}{NC} = 1 + \frac{C\Omega''}{NC}.$$

Multiply numerator and denominator of the last quotient by $C\Omega''$. Now the product of NC by $C\Omega''$ is equal to the square of CM, where M is the intersection of $C\Omega''$ with $\bar{q}$. Hence

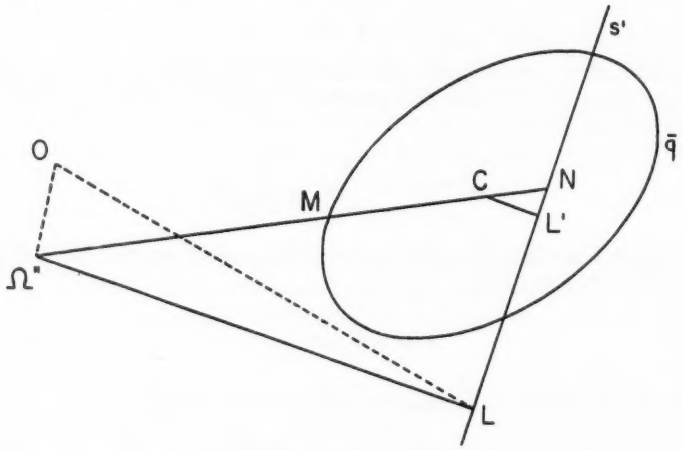
$$\frac{N\Omega''}{NC} = 1 + \frac{(C\Omega'')^2}{(CM)^2} = \frac{(CM)^2 + (C\Omega'')^2}{(CM)^2}.$$

If R and r denote the lengths of the segments CM and $C\Omega''$ respectively, formula (14) takes the form

$$\mathbf{S}\cdot\boldsymbol{\omega} = \frac{\Delta}{\delta}\,\omega''^2\,\frac{R^2+r^2}{R^2}. \tag{15}$$

This formula requires only the construction of $\boldsymbol{\omega}''$ and $\bar{q}$ for the calculation of the scalar product $\mathbf{S}\cdot\boldsymbol{\omega}$.

If $\boldsymbol{\omega}$ represents the instantaneous rotation of a rigid body with a fixed point O, the kinetic energy, equal to one half of $\mathbf{S}\cdot\boldsymbol{\omega}$, can be graphically determined if the six components of the tensor of inertia attached to O are known. Moreover, dividing this scalar product by ω^2 we obtain the moment of inertia I of the body



with respect to the instantaneous axis of rotation. As the quotient of ω'' by ω is equal to $\cos\gamma$, where γ is the angle formed by $\boldsymbol{\omega}$ and OZ, formula (15) yields

$$I = \frac{\Delta}{\delta}\cos^2\gamma\,\frac{R^2+r^2}{R^2}. \tag{16}$$

Denoting by ρ^2 the quotient of Δ by $m\delta$ where m is the mass of the body, we have the following expression for the radius of gyration k corresponding to the direction of $\boldsymbol{\omega}$:

$$k = \rho\cos\gamma\,\frac{\sqrt{R^2+r^2}}{R}.$$

The graphical construction of k is easy, once the ellipse $\bar{q}$ has been drawn.

The graphical methods developed in this paper will be useful in a graphical discussion of the rotation of a rigid body around a fixed point.

A PROJECTIVE CONSTRUCTION FOR PLANE NODAL CUBICS

I. C. FISCHER, University of Minnesota

1. The problem. The problem under consideration is to discover the base elements for the construction of plane nodal cubic curves, with the use of straight edge alone, having given a sufficient number of conditions for the unique generation by a projective method. A few of the many possible cases which may arise will be discussed here; others can easily be formulated from the general construction.

In another paper,* the author presented a method for constructing a plane sextic, with degenerations, in which the base elements were a point conic, two straight lines, and three mutually perspective flat pencils. If the conic is replaced by a straight line, a cubic of the class with which this paper is concerned will result.

The method of construction employed here somewhat resembles Grassmann's,† in that three fixed points and three fixed lines are employed, but the method of generation is entirely different since we use the three fixed points as centers for three mutually perspective flat pencils. Another straight line construction‡ utilizes two pencils of lines in one-two correspondence. The author's solution is obviously a special case of a more general construction referred to above.‡

2. The general construction. Consider three noncollinear points A, B, C, and the lines l, l', l'', which are not concurrent. A ray through A cuts l in point L. The rays BL and CL meet l' and l'' in L' and L'' respectively. Line $L'L''$ is on a point P of AL. The totality of all points P so defined constitutes the required locus. Before proving that this locus is a nodal cubic, a few projective relations will be discussed.

Let $[L]$ be a range of points on l, and $[a]$ a pencil of lines on point A; also let $\overset{O}{\asymp}$ and $\asymp$ represent a perspective correspondence with center O, and a projectivity, respectively. The chain of perspectives

$$[L'] \underset{\asymp}{\overset{B}{}} [L] \underset{\asymp}{\overset{C}{}} [L'']$$

defines a projective correspondence Π such that $\Pi[L'] = [L'']$, and the lines $L'L''$ defined by all pairs of homologous points on the two ranges generate a line conic k, of which l' and l'' are elements. Since $[L']$ is perspective with $[b]$ then $[b]$ is perspective with k and

$$[a] \underset{\asymp}{\overset{l}{}} [b] \underset{\asymp}{\overset{l'}{}} k; \quad \text{also} \quad [a] \underset{\asymp}{\overset{l}{}} [c] \underset{\asymp}{\overset{l''}{}} k.$$

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* Projective construction for certain algebraic curves, this MONTHLY, April 1940.

† Plane Curves of the Third Order, H. S. White, p. 108, 1925.

‡ On the straight line construction of unicursal cubics, W. H. Bunch, this MONTHLY, Vol. 42, 1935, pp. 74–80.

Therefore, $[a] \asymp k$, and since homologous pairs of $[a]$ and k intersect in the points P, then P is the locus of the intersection of homologous lines of a line conic and a projective pencil of lines. That all the points P define a cubic uniquely will be shown by a synthetic and an analytic proof.

2-a. *A synthetic proof.* Consider a point conic K with M any point on it such that the pencil $[m]$ is perspective with K and also projective with another pencil $[a]$ with center A not on K. Then $[a]$ and $[m]$ generate a point conic K' which is projective with K. Thus A is on K' and M is an intersection of K and K'. To each point on K there corresponds one and only one point on K' and such homologous pairs coincide only in the intersections of the two conics. There are at most four points common to K and K', let us say at M, M_1, M_2, M_3. Rays AM_1, AM_2, AM_3, are homologous to rays MM_1, MM_2, MM_3. The ray on M which corresponds to AM on A is a tangent to K' and the point on K homologous to M on K' does not coincide with M. Therefore, there are at most three lines of $[a]$ which pass through their corresponding points on K.

A polarity with respect to K transforms the points of K into a line conic k, and the rays of $[a]$ into a range of points $[A']$ on the polar of A. Thus $[a]$ is projective with k, and $[A']$ is projective with k. It follows that not more than three points of $[A']$ will lie on their associate lines of k and these three points are on the locus of P. Now consider any line of the plane such that its points are in projective correspondence with $[A']$. There are then at most three points of this line on the homologous lines of k and these common points are also on the required locus. Therefore, the locus is a point cubic.* That the curve is nodal is shown by the fact that the polar of A with respect to K cuts K in two distinct real points or in no real points, since A is not on K. These intersections are double points of the polar transformation. Hence two distinct lines of k or no such lines pass through A and are intersected in A by homologous rays of $[a]$. Therefore A is an acnode or crunode and is the only possible singularity of the cubic, since a straight line intersects a cubic in at most three real points.

2-b. *The equation of the cubic.* The analytic proof that the construction yields a cubic is conveniently demonstrated by the use of homogeneous point coordinates (x_1, x_2, x_3), with parameters in non-homogeneous form. In what follows it will be understood that the symbol a_ix_i represents a summation with $i = 1, 2, 3$.

Let the vertices of the triangle of reference be on the points A, B, C, and let these points be defined by the homogeneous point coordinates (0, 0, 1), (1, 0, 0), and (0, 1, 0) respectively. The various base elements are given by the following equations:

(1) $\qquad a_ix_i = 0$, the line $l = 0$;

(2) $\qquad b_ix_i = 0$, the line $l' = 0$;

(3) $\qquad c_ix_i = 0$, the line $l'' = 0$;

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* See problem 7, p. 217, Projective Geometry, Veblen and Young, Vol. 1.

(4) $x_1 - hx_2 = 0$, the pencil $[a]$ on A;

(5) $x_2 - mx_3 = 0$, the pencil $[b]$ on B;

(6) $x_1 - nx_3 = 0$, the pencil $[c]$ on C.

The ranges $[L']$ and $[L'']$, in terms of the parameters are:

(7) $L'[-(b_2m + b_3), b_1m, b_1]$ from (2) and (5);

(8) $L''[nc_2, -(c_1n + c_3), c_2]$ from (3) and (6).

The equation of the line conic k in parametric point coordinates, may be expressed in the usual determinant form:

(9) $$\left| x_i, L', L'' \right| = 0.$$

Now since $[a]$, $[b]$, $[c]$, are mutually perspective on axis l, that is, homologous rays of the three pencils are concurrent, the determinant of the coefficients of (4), (5), (6), vanishes, and this fact gives the relation among the parameters:

(10) $$n = hm.$$

The projective correspondence which brings L' into L'', also transforms m into n. Since for each point P corresponding rays of $[b]$ and $[c]$ meet on l, it follows that

(11) $a_1n + a_2m + a_3 = 0$, from (1), (5), and (6).

(12) $(a_1h + a_2)m + a_3 = 0$, from (10), (11).

Solving for m out of (12), and noting that $h = x_1/x_2$ as given by (4), we obtain

(13) $$m = -\frac{a_3x_2}{p}, \quad \text{where } p \text{ is defined below.}$$

(14) $$n = -\frac{a_3x_1}{p}, \quad \text{using (10) and (13).}$$

Now substituting (13) and (14) into (9) and expanding, we obtain, after grouping terms, the equation of a cubic:

(15) $$f(x_1, x_2, x_3) = 0, \text{ or}$$

$$p^2 s - a_3 pr + a a_3^2 t = 0, \text{ where}$$

$$p = a_1x_1 + a_2x_2,$$

$$r = b_1c_1x_1^2 + b_2c_2x_2^2 + 2b_1c_2x_1x_2 + b_3c_1x_1x_3 + b_2c_3x_2x_3,$$

$$s = b_1c_3x_1 + b_3c_2x_2 + b_3c_3x_3,$$

$$t = x_1x_2x_3, \qquad a = b_2c_1 - b_1c_2.$$

The general cubic has nine essential constants, whereas the equation (15) has but six such coefficients. This indicates a cubic with specified node. If the

node is not specified by the particular orientation of the base elements given above, the most general equation of $f=0$ contains eight essential constants, which shows that (15) is a nodal cubic.

2-c. *Some fundamental points on f.* Certain points of f can be easily found from the construction and equation. These will be of use in the problems to follow. See Fig. 1.

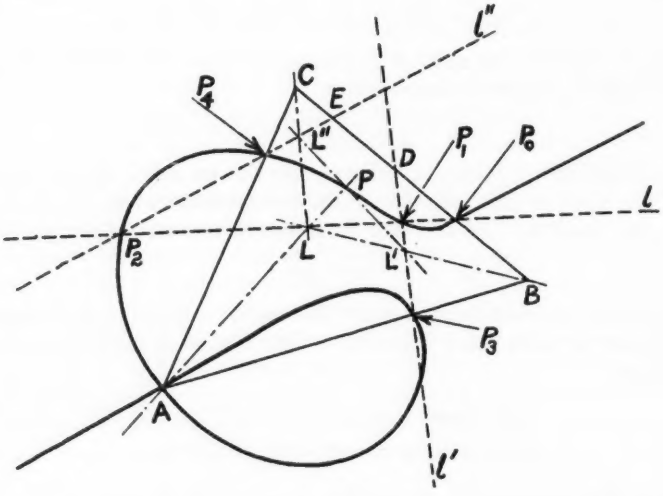


FIG. 1

P_1 is the intersection of l and l', denoted by l, l'; P_2 is on l, l'' and P_3 on $l', AB; l'', AC$ intersect on P_4, and l, BC on P_0. By means of (15) it is easily shown that $p=0$ is the line through A and P_0; $s=0$ is a line P_3P_4; $r=0$ a conic on A, P_3, P_4; $t=0$ is the improper cubic composed of the three axes of the reference triangle.

That A is the node can be shown by the fact that any line through A has two coincident intersections with f. It is not difficult to prove that all the derivatives $\partial f/\partial x_i$ vanish at A, and therefore A is the double point.

3. The inverse of the general construction. The methods for finding the base elements for the construction of f under specified conditions can now be tabulated. To make the procedure as general as possible, the given simple points will be chosen so that no three are collinear, although this restriction is not necessary. The specification of three collinear points in certain order may cause the construction to break down.

Since the cubic is nodal, eight ordinary points or equivalent conditions may be specified to determine it uniquely. If the node is specified, six additional conditions must be imposed. Thus, in addition to the fundamental points men-

tioned, use will be made of P_5 which is located by the ray joining A with l, P_3P_4, and P_6 on a ray through A and P_4P_5, l. P_7 and P_8 are on rays AD and AE, where D and E are the junctions of BC, l' and BC, l'' respectively. The following constructions are readily understood when the general construction is retraced in the reverse order.

3-a. *Given points* P_n ($n=1$–8 *inclusive*). P_1P_2, P_1P_3, P_2P_4 are taken as l, l', l'' respectively. P_3P_4 and P_4P_5 cut l in L_5 and L_6. Then P_5L_5 and P_6L_6 meet in A. AP_7 and AP_8 cut l' and l'' in D and E. DE will intersect AP_3 in B and AP_4 in C, and the construction for any number of points on the curve is uniquely determined.

3-b. *Given* P_n (*with* $n=7$ *deleted*), *and tangent* p_1 *at point* P_1. The tangent p_1 is the line P_1Q_1, where AB and l intersect in L_1. and AC cuts L_1D in Q_1. This result was not obtained synthetically but while working on an analytical phase of the problem. The author had on hand the analytical expressons for many points and lines of the construction figure including the equations of certain tangents. By comparing coefficients, the above construction was arrived at. Thus the method of solution was tentative at best. The analytical verification is a little tedious, but not at all difficult. If P_1 is taken on (1, 1, 1) there will be no loss in generality and we then have

$$a_1 + a_2 + a_3 = 0 \quad \text{and} \quad b_1 + b_2 + b_3 = 0,$$

equations of l and l' may be written

$$a_1x_1 - (a_1 + a_3)x_2 + a_3x_3 = 0, \text{ and}$$
$$b_1x_1 + b_2x_2 - (b_1 + b_2)x_3 = 0, \text{ respectively.}$$

The equation of p_1 as constructed above is then

$$(a_1b_2 + a_3b_1)x_1 - a_1b_2x_2 - a_3b_1x_3 = 0.$$

That this is the equation of the tangent to f at P_1 may be shown by the general equation for a tangent:

$$x_i \frac{\partial f}{\partial x_i} = 0,$$

where the partial derivatives are evaluated for the point P_1, or (1, 1, 1). This will be relatively simple if (15) is differentiated as given in its unexpanded form and subsituting coefficients $a_2 = -(a_1+a_3)$ and $b_3 = -(b_1+b_2)$, in the final reduction.

Base lines l, l', l'', and point A are located as in (3-a). AP_3 and AP_4 meet l and p_1 in L_1 and Q_1 respectively. Q_1L_1 cuts l' in D and AP_3 cuts l'' in E. Then DE meets AL_1 in B and AQ_1 in C, and the construction is complete. The process, however, may be simplified by the fact that l' is the tangent at P_1 if C is on l'. In this case C coincides with D.

3-c. *Given P_n ($n=1$–6 inclusive), and p_1, p_2 the tangents on P_1 and P_2 respectively.* The tangent p_2 was arrived at directly by associating P_2 with P_1 in the projective relations involved. By a process similar to that used in 3-b it may be shown that p_2 is the line Q_2P_2, where l cuts AC in L_2 and EL_2 meets AB in Q_2.

A, l, l', l'', are located as in 3-a and 3-b. Q_1L_1 and Q_2L_2 meet l' and l'' in D and E respectively. ED then determines B and C. This construction may also be specialized by taking B and C on l'' and l', in which case p_1 and p_2 will coincide with l' and l'' respectively.

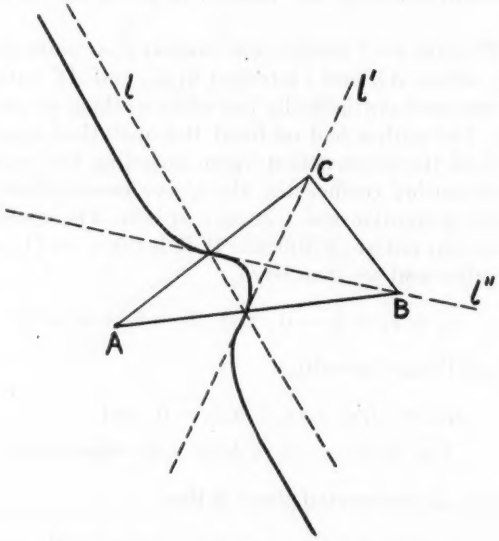


FIG. 2

3-d. *Given the node A and six simple points P_n ($n=1, 2, 3, 4, 7, 8$).* P_1P_2, P_1P_3, P_2P_4, will determine l, l', l'', respectively as before. AP_7 cuts l' in D and AP_8 meets l'' in E. DE then will intersect AP_3 in B and AP_4 in C.

3-e. *Given the node A with real tangents a' and a'', and four ordinary points* P_1, P_2, P_3, P_4. It will be noted that the two lines of the conic k which pass through A determine the real tangents on the node. Let the points P_1, P_2, P_3, P_4, locate the lines l, l', l'', as in the previous cases. Let A be the intersection of any two lines of k, these cutting l' in L_1' and L_1'', and l'' in L_2' and L_2'' respectively. The given tangents a' and a'' cut l in M' and M''. The junction of $L_1'M'$ with $L_1''M''$ is point B. In a similar manner $L_2'M'$ and $L_2''M''$ meet in point C.

It is easily shown that if l' and l'' intersect on A, then AB and AC are real nodal tangents. This fact makes it possible to simplify the construction. In this case let the distinct given points be P_1, P_2, P', P'', where P' and P'' are any two points other than those designated as fundamental. P_1P_2, AP_1, AP_2, deter-

mine l, l', l'' respectively. $P'A$ cuts l in Q'; $Q'P''$ meets l' in L' and l'' in L''. $P''A$ cuts l in Q''; then $Q''L'$ meets a' on B and $Q''L''$ intersects a'' on C.

This case is evidently subject to variation, since when l' is on A, then AB is a tangent on A. Also if l'' is on A, then AC is a tangent on A.

4. Miscellaneous. The node may be made an isolated point by the specification of two real inflections, or by special orientation of the base elements. If l, l', and AB are concurrent, then l' is a tangent on P_1; similarly, if l, l'', and AC are concurrent, l'' is tangent on P_2. If both of these conditions are imposed, the double point A will be, in general, isolated (see Fig. 2). Now if to these restrictions we add the conditions that l' is on C and l'' on B, it will be found that the Hessian of f vanishes at P_1 and at P_2. Thus, P_1 and P_2 are inflections and l', l'', the corresponding inflectional tangents. Many other specializations, exist.

Constructions for conics will arise from the foregoing problems if the cubic is degenerate. If the base lines are chosen so that l, l', AB; l, l'', AC; and l', l'', BC, are three points, the cubic will break off into the line $p=0$ and a conic through A, P_1, P_2, with tangents l' and l'' on P_1 and P_2 respectively. Thus point l', l'' will be a pole and the line $s=0$ its polar with respect to the conic. The equation of this conic is $ps-a_3r=0$.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A NOTE ON THE RIEMANN INTEGRAL

H. E. ROBBINS, Post Graduate School, U. S. Naval Academy

The following theorem, which arose during an investigation of integration by change of variable, is believed to provide an interesting generalization of the ordinary definition of the Riemann integral of a continuous function.

THEOREM. *Let $f(x)$ be a continuous function and let C be a constant given in advance. Choose a sequence of numbers (not necessarily increasing)*

$$a = x_0, x_1, x_2, \cdots, x_n = b$$

such that

$$\sum |x_i - x_{i-1}| < C.$$

Set $\max |x_i - x_{i-1}| = \delta_n$, *and form the sum*

$$S_n = \sum f(\bar{x}_i)(x_i - x_{i-1}),$$

where $\bar{x}_i$ is any point in the closed interval whose endpoints are x_i and x_{i-1}. Then for

any sequence of such sums in which $\delta_n \to 0$, we have

$$\lim S_n = \int_a^b f(x)dx,$$

where the integral on the right is the ordinary Riemann integral.

Remark. In this theorem C may be arbitrarily large but must be fixed for all sums in the sequence. Without this restriction the theorem is false, as may easily be shown.

We proceed to prove the theorem. Let the maximum oscillation of $f(x)$ in any interval of the subdivision be equal to ϵ. Consider the set of all points

$$a = y_0, y_1, y_2, \cdots, y_m = b,$$

consisting of the x_i arranged in increasing magnitude without repetition. In each interval $[y_{k-1}, y_k]$ choose a definite point $\bar{y}_k$. Suppose the interval $[x_{i-1}, x_i]$ contains the points $y_{k-1}, y_k, y_{k+1}, \cdots, y_{k+r}$ as endpoints or interior points. Replace the term

$$f(\bar{x}_i)(x_i - x_{i-1}) \tag{1}$$

by the expression

$$f(\bar{y}_k)(y_k - y_{k-1}) + \cdots + f(\bar{y}_{k+r})(y_{k+r} - y_{k+r-1}) \tag{2}$$

if $x_i > x_{i-1}$, and by the negative of this expression if $x_i < x_{i-1}$. The difference between the term (1) and the corresponding expression (2) will be less in magnitude than $\epsilon|x_i - x_{i-1}|$. Carrying out the replacement for every term of S_n, we obtain a new sum S_n' differing from S_n by less than

$$\epsilon \sum |x_i - x_{i-1}| < C\epsilon.$$

In S_n' collect all terms

$$\pm f(\bar{y}_k)(y_k - y_{k-1})$$

for a fixed k. The result will be simply

$$f(\bar{y}_k)(y_k - y_{k-1}),$$

since the interval $[y_{k-1}, y_k]$ will be traversed some number p times from left to right and $(p-1)$ times from right to left. Since this is true for all k, S_n' is an ordinary Riemann sum for $f(x)$, and hence

$$\lim S_n' = \int_a^b f(x)dx.$$

But

$$\lim (S_n - S_n') = 0,$$

since $|S_n - S_n'| < C\epsilon$ and $\lim \epsilon = 0$. This completes the proof.

AN EXAMPLE ON DOUBLE SERIES

J. E. BROCK, Washington University

It is known that the convergence of a double series $\sum_{i=1,j=1}^{\infty,\infty} a_{ij}$ *i.e.*, the existence of the double limit, $\lim_{m\to\infty,n\to\infty} S_{mn}$ (where $S_{mn}=\sum_{i=1,j=1}^{m,n} a_{ij}$), does not imply the convergence of the series $\sum_{i=1}^{\infty} a_{ij}$ or $\sum_{j=1}^{\infty} a_{ij}$. Bromwich* cites an example of a convergent double series, a finite number of whose rows and columns diverge. It might be of interest, particularly for instructional purposes, to have a simple example in which the series $\sum_{i=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty} a_{ij}$ diverge for *all* j and i.

Consider the double series $\sum_{i=1,j=1}^{\infty,\infty} a_{ij}$ where $a_{ij}=(-1)^i b_i+(-1)^j b_j$ and $\sum_{n=1}^{\infty} b_n$ converges and has the sum 0. It may easily be seen that the double limit exists, for the partial sum $\sum_{i=1,j=1}^{m,n} a_{ij}=\sum_{i=1,j=1}^{m,n}(-1)^i b_i+\sum_{i=1,j=1}^{m,n}(-1)^j b_j$ and the sums on the right are either equal to zero or to $-\sum_{i=1}^{m} b_i$ or to $-\sum_{j=1}^{n} b_j$. Since $\sum_{n=1}^{\infty} b_n=0$, these latter sums approach zero as a limit as $m\to\infty$, $n\to\infty$.

However, $\sum_{i=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty} a_{ij}$ both diverge, since the general term a_{ij} does not approach zero as a limit, when for fixed i, $j\to\infty$, or when for fixed j, $i\to\infty$.

FUNCTIONS NOT FORMULAS FOR PRIMES

IRVING REINER, Brooklyn, New York

A function $F(x)$ is said to be a formula for primes if $F(x)$ is a prime for every positive integral value of x. We shall show that certain types of functions cannot be formulas for primes.

LEMMA. *If $f(x)$ and $g(x)$ are polynomials with integral coefficients and positive leading coefficients, then for all sufficiently large integral values of k*

$$f(a+kp(p-1))^{g(a+kp(p-1))} \equiv f(a)^{g(a)} \pmod{p},$$

where p is any prime and a is any integer such that $g(a)>0$.

Proof. We have $f(a+kp(p-1))\equiv f(a) \pmod{p}$, and $g(a+kp(p-1))\equiv g(a) \pmod{p-1}$. Therefore we may write

$$f(a+kp(p-1))^{g(a+kp(p-1))} \equiv f(a)^{g(a)+M(p-1)} \pmod{p},$$

where M is positive for sufficiently large k. However, if $f(a)\equiv 0 \pmod{p}$, then $f(a)^{M(p-1)}\equiv 0 \pmod{p}$, and if $f(a)\not\equiv 0 \pmod{p}$, then $f(a)^{M(p-1)}\equiv 1 \pmod{p}$ by Fermat's theorem. In either case

$$f(a+kp(p-1))^{g(a)+M(p-1)} \equiv f(a)^{g(a)} \pmod{p},$$

and the lemma is proved.

THEOREM 1. *If $f_i(x)$, $g_i(x)$ $(i=1, \cdots, n)$ are polynomials with integral coefficients and positive leading coefficients, the following is not a formula for primes:*

$$F(x) = \sum_{i=1}^{n} f_i(x)^{g_i(x)}.$$

* Bromwich, T. J. I'A., Theory of Infinite Series, Macmillan (1908), p. 90.

Proof. Otherwise, suppose that $F(x)$ is a formula for primes; let a be an integer such that all of the $g_i(a)$ are positive. Then we have for sufficiently large integral k (by the Lemma)

$$f_i(a + kp(p-1))^{g_i(a+kp(p-1))} \equiv f_i(a)^{g_i(a)} \pmod{p},$$

where $p = F(a)$. Hence we have

$$F(a + kp(p-1)) \equiv \sum_{i=1}^{n} f_i(a)^{g_i(a)} \pmod{p}.$$

Therefore $F(a+kp(p-1))$ is divisible by p; but we assumed that $F(x)$ is prime for all positive integral x. Hence $F(a+kp(p-1))$ is either p or $-p$ for all sufficiently great k. This contradicts the fact that $\lim_{x\to\infty} F(x) = \infty$, which is true because the leading coefficients are positive. Hence, $F(x)$ is not a formula for primes.

Example. If a and b are integers, $a>1$, then a^x+b is not a formula for primes.

THEOREM 2. *Let $F(x)$ be defined as*

$$F(x) = \sum_{i=1}^{n} f_i(x)^{[g_i(x)^{h_i(x)}]},$$

where $f_i(x)$, $g_i(x)$, $h_i(x)$ $(i=1, \cdots, n)$ are polynomials with integral coefficients and positive leading coefficients. Then $F(x)$ is not a formula for primes provided there exists an integer a such that:

(1) *All the $g_i(a)$ and $h_i(a)$ are positive;*

(2) *If any $g_i(a)$ has a factor r_i in common with $q = F(a) - 1 = p - 1$, then any prime divisor of r_i occurs as a factor of q to a power less than or equal to $h_i(a)$.*

Proof. Assume that $F(x)$ is a formula for primes, and let a be an integer whose existence is demanded by the hypothesis. We shall use the above notation. Let the prime factorizations of $q = q_1^{u_1} q_2^{u_2} \cdots q_s^{u_s}$ and $r_i = q_1^{v_1} q_2^{v_2} \cdots q_t^{v_t}$, where $t \leqq s$ and each $v_i > 0$. Clearly this places no restriction on the q_i. Choose $m_i = M_i q \phi(q)$, where $\phi(q)$ is the Euler ϕ-function, such that $g_i(a+m_i)$ and $h_i(a+m_i)$ are positive. Now

$$g_i(a + m_i)^{h_i(a+m_i)} \equiv g_i(a)^{h_i(a)+M'\phi(q)} \pmod{q},$$

where M' is positive for proper choice of M_i. Let $g_i(a) = r_i t_i$, where t_i is prime to q. Then since $h_i(a) \geqq \max(u_1, u_2, \cdots, u_t)$ and since the Euler function is $\phi(q) = \phi(q_1^{u_1} \cdots q_t^{u_t})\ \phi(q_{t+1}^{u_{t+1}} \cdots q_s^{u_s})$, we have

$$r_i^{M'\phi(q)+h_i(a)} \equiv r_i^{h_i(a)} \pmod{q}.$$

Also $t_i^{\phi(q)} \equiv 1 \pmod{q}$. Therefore

$$g_i(a + m_i)^{h_i(a+m_i)} \equiv g_i(a)^{h_i(a)} \pmod{q}.$$

Now let $m = Mpq\phi(q)$. Then by the reasoning of Theorem 1, for all sufficiently

large values of M, $F(a+m)$ equals p or $-p$. Hence, we again have a contradiction as in Theorem 1, and $F(x)$ is not a formula for primes.

Example 1. $2^{2^x}+k$ is not a formula for primes for integral k.

Example 2. $F(x)=(x+1)^{x^x}+1$ is not a formula for primes. Proof, let $a=1$.

Note. This method is easily extended to functions of the type

$$f(x)^{g(x)^{h(x)^{k(x)}}}$$

under conditions similar to those of Theorem 2.

ON P_r-MATRICES

H. SCHWERDTFEGER, University of Adelaide

The notion of P_r-matrix which has been introduced in a preceding note* can further be restricted usefully in the following way when it still covers all ϕ-symmetric matrices: Let again ϕ be an involutory automorphism or antimorphism of the infinite field $\mathfrak{F}$ containing the matrix elements. For any α in $\mathfrak{F}$ the element $\phi(1)\cdot\phi(\alpha)$ may be called the ϕ-conjugate of α. An n-rowed square matrix A with elements in $\mathfrak{F}$ is said to be a ϕP_r-matrix if it is a P_r-matrix ($r\leqq n$) and the linear relations existing between its rows arise from those between the equally numerated columns by taking the ϕ-conjugate elements of the coefficients of the column relations in corresponding places. Thus when $a^{(i_1)}, \cdots, a^{(i_r)}$ are linearly independent columns of A, then $a'_{(i_1)}, \cdots, a'_{(i_r)}$ are linearly independent rows of A, and from

$$a^{(\nu)} = \sum_{\rho=1}^{r} \lambda_\rho^\nu a^{(i_\rho)} \qquad (\nu = 1, \cdots, n)$$

follows

$$a'_{(\nu)} = \phi(1)\cdot \sum_{\rho=1}^{r} \phi(\lambda_\rho^\nu) a'_{(i_\rho)}.$$

If $r=n$ all λ_ρ^ν are zero except for $i_\rho=\nu$. Thus every regular n-matrix in $\mathfrak{F}$ is a ϕP_n-matrix. A singular ϕP_r-matrix ($r<n$), however, is ϕ-congruent with

$$\begin{pmatrix} A_r & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

where A_r is an r-rowed principal submatrix of rank r in A. Here A_r can be any regular r-matrix. Hence the ϕP_r-matrices form the most general class of matrices A of rank r which are ϕ-congruent with a matrix of type (1) where A_r is any regular principal submatrix of A.

* H. Schwerdtfeger, On generalized hermitian matrices, this MONTHLY, vol. 49, 1942, pp. 181–184. The content of §§1–2 of this note is supposed to be known here. (On p. 182, in the last formula, A_ϕ^2 has to be replaced by $A_{\phi\mathfrak{P}}$.)

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

SOLVING A RIGHT TRIANGLE WITHOUT TABLES

J. S. FRAME, Michigan State College

Given the three sides a, b, c of a right triangle ABC, in which $C=90°$ and $a<b$, it is possible to express the smallest angle A in terms of the three sides by the approximate formula

$$A \cong \left(\frac{a}{b+2c}\right)(172°), \tag{1}$$

in which the error is less than one minute for angles less than 37°, and is still fairly small for all angles less than 45°, as the following table shows:

(2)

A in degrees	0°	10°	20°	30°	33°	36°	39°	42°	45°
Error in minutes	0′	+.4′	+.7′	+.4′	+.04′	−.5′	−1.4′	−2.7′	−4.4′

For example, to approximate the smallest angle in the familiar 3, 4, 5 triangle, we should have

$$A = \left(\frac{3}{14}\right)(172°) = \frac{258°}{7} = 36°51.4'. \tag{3}$$

The true value from tables is $A=36°52.2'$. Thus the accuracy is better than could be obtained from a slide rule, and does not require trigonometric scales.

The following considerations led to the discovery of this approximate formula. It can be seen, either from tables or from a consideration of power series, that the radian measure of a small angle lies approximately one-third of the way from the sine to the tangent. However, this approximation is obviously not valid for angles near 90°, since the tangent becomes infinite. Suppose therefore that we try to use not the arithmetic but the harmonic average of $\sin\theta$ and $\tan\theta$, weighted two to one, and see if it is a good approximation to the radian measure of an angle θ. We then have

$$\frac{3}{\theta \text{ (radians)}} = \frac{1}{\sin\theta} + \frac{1}{\sin\theta} + \frac{1}{\tan\theta}. \tag{4}$$

For $\theta=\pi/6$ radians, we have the approximation

$$18/\pi = 5.732, \text{ instead of the true value } 5.730, \tag{4a}$$

whereas for $\theta=\pi/2$ radians, we have

(4b) $6/\pi = 2$, instead of the true value 1.910.

If we let $A = (\theta/\pi)\ 180°$, so that A is measured in degrees, and if we write $\sin A = a/c$, $\tan A = a/b$, then formula (4) becomes

$$(5) \qquad \frac{540°}{\pi A \text{ (degrees)}} = \frac{c}{a} + \frac{c}{a} + \frac{b}{a} = \frac{b + 2c}{a}$$

or

$$(6) \qquad A \text{ (degrees)} = \left(\frac{a}{b + 2c}\right)\frac{540°}{\pi} = \left(\frac{a}{b + 2c}\right)(171.88°).$$

This formula is extremely accurate for angles less than 20°, but always gives too small a value. We therefore increase the numerical coefficient slightly from 171.88° to 172°. This is easier to work with numerically, and gives a better fit above 30°, without greatly increasing the error for angles less than 30°. Thus we arrive at formula (1).

The correction for formula (1) which is the negative of the error in table (2) is very closely given by the expression:

(7) **Correction for formula (1) in minutes** $= -1.2'(A/30°) + 0.8'(A/30°)$.[5]

It is suggested that this formula (1) might be used to advantage on slide rules having no trigonometric scales, for the purpose of solving right triangles when two sides are known, if the third side is first obtained by the Pythagorean theorem. (See this MONTHLY, vol. 50, p. 55.)

CLUB REPORTS 1942-43

Harvard Mathematical Club, Harvard University

The following talks were presented at the meetings of the club in 1942–43:
The reversibility paradox of hydrodynamics, by Professor Garrett Birkhoff.
Picturing surfaces in four dimensions, by Professor Hassler Whitney.
Mathematics in Mexico and South America, by Professor G. D. Birkhoff.
A proof of Stone's representation theorem, by Leon Kruger.
Construction of measures, by Dr. Edwin Hewitt.
Carrolean Sorites, by Robert Buck.
A random series, by Dr. Harry Pollard.
Projective geometry, by Lindley Burton.
Graduating statistical series, by Robert Hoskins.
Groups as algebras of a single operation, by Jeremiah Certaine.
Hilbert's seventh problem, by Michael Norris.
Lebesgue integration, by Matthew Gaffney, Jr.

Recipients of the Robert Fletcher Rogers Prizes for the best student talks were Mr. Gaffney ($35) and Mr. Burton ($15). Officers for 1942–43 were: President, Michael J. Norris; Vice-President, Robert H. Hoskins; Secretary, Joseph A. Zilber; Treasurer, Lindley Burton; Faculty Adviser, Professor G. D. Birkhoff.

Mathematics Club, University of Kansas

The Club's meetings for the year 1942–43 were divided into two parts. During the first semester the meetings featured talks by faculty members, while during the second semester the speeches were given by undergraduates. The annual fall picnic opened the Club's program, which ended with the annual spring picnic at which prizes for the best speeches were awarded and the officers for the coming year were announced. Meetings were held every two weeks during the school year. The faculty speeches were:

Navigation, by Professor N. W. Storer.
Cryptography, by Professor G. W. Smith.
Probability in artillery fire, by Captain E. E. Baker, CAC.
Some strange phenomena of quantum mechanics, by Dr. L. N. Liebermann.
A problem in convex sets, by Professor G. B. Price.

The undergraduate speeches included the following:

Some problems in naval gunnery, by Harwood Kolsky.
Squaring a circle, doubling a cube, trisecting an angle, by Jean Bartz.
Electricity and imaginary numbers, by Howard Barnett.
Curve fitting, by John Yarnell.
Ancient mathematical puzzles, by Mary Steel.
A mathematical approach to physics, by Rachel Ragel.

Prizes were offered by members of the mathematics department to the three best student speakers. To Harwood Kolsky was awarded the first prize: a copy of *Men of Mathematics* by E. T. Bell, donated by Dean E. B. Stouffer. To Howard Barnett went the second prize: a copy of *What is Mathematics* by Courant and Robbins, donated by Professor J. J. Wheeler. To Mary Steel was given the third prize: a copy of *Amusements in Mathematics* by H. E. Dudeney, donated by Professor P. O. Bell. The officers for this past year have been: President, Harwood Kolsky; Vice-President, Jean Bartz; Secretary-Treasurer, John Yarnell; Social Chairman, Virginia Stephenson and Rachel Ragel; Faculty Adviser, Professor P. O. Bell.

Mathematics Club, Iowa State College

The Club held seven meetings during the school year.

Integral Calculus, a movie, showed pictorially how volume enclosed by surfaces is found by integration, and was a valuable aid in clarifying the integration process for the student of integral calculus.

The operation of statistical machines was the subject of a demonstration meeting which focused attention on the place of statistics in a nation at war. At several of the meetings word problems were given which illustrated the topics discussed. Those present participated actively in solving these problems. The meetings were well attended. The Program Committee consisted of Glee Barth, Joe Elliot, William Slaichert, Eleanor Hoeflin and Marian Carlin. The Faculty Adviser was Mr. Fred Robertson.

Pi Mu Epsilon, University of Nebraska

Less leisure time than usual for most of the students resulted in fewer meetings of our chapter this year. Activities for the year included three program meetings, a social meeting, the annual initiation banquet, and a spring picnic and initiation. Following the usual custom, competitive examinations in analytical geometry and in calculus were held. The winners, Walter Koenig and Norman Zabel, were each awarded ten dollars.

At the beginning of the second semester, a new plan was started to stimulate interest in mathematics. Each week five problems of varying difficulty were proposed for solution. These problems were posted on the bulletin board under their proper heading: *i.e.*, calculus, analytics, trigonometry, algebra, or general. The solutions were discussed at the regular meetings.

Programs included the following topics:

La Place transformation, by Lloyd Jackson.

The solution of problems in electrical circuits and transmission lines by the La Place transformation, by J. R. Parker.

The Hatchet planimeter, by Kotaro Murai.

Officers were as follows: Director, Herman Krueger; Secretary, Marcia Beckman; Treasurer, William Ruyle.

Mathematics Club, Wayne University

The Mathematics Club at Wayne University held regular meetings throughout the year on the average of one every two weeks. Its aim is to assemble all students who are interested in mathematics so that they can get acquainted. This was accomplished by our meetings which consisted of academic papers and social affairs. The outstanding topics discussed were:

The impossibility of the trisection of an angle, by Jane Cronin.

Courses in mathematics, by Dr. M. Coral.

Mathematics in map making, by Robert Walton.

Casting out nines, by Aline Hochberg.

The cube root of a number, by Leonard Antel.

On May 23, 1943, we held our annual picnic to which the mathematics professors and their families were invited. A sack race was held with professors and students competing for a pound of coffee—a student won. With too much to eat and lots of fun, the picnic ended with everyone sitting around a campfire singing college songs. Besides the annual picnic, the Club had a Christmas tea, an informal evening at the home of the faculty adviser, and an Easter party at the home of the new president. The officers for 1942–43 were: President, Evelyn Kurz; Vice-President, Robert Walton; Secretary-Treasurer, Myrtle Keryluk. Those for 1943 are: President, Myrtle Keryluk; Vice-President, Dick Kuba; Secretary-Treasurer, Aline Hochberg; Publicity Chairman, Natalie Costrell; Faculty Adviser, Dr. K. W. Folley.

Pi Mu Epsilon, University of Wisconsin

A lantern demonstration of plane and twisted curves in space was presented by Dr. Florence E. Allen and Professor Paul L. Trump at the opening meeting. Later meetings included the following talks:

Use of matrices in solving linear equations, by Dr. R. H. Brucks.

Mathemotical problems pertaining to the unsteady state of heat flow, by Mr. George Thodos.

Boolean algebra, by Mr. Robert Simpson.

The theory of counting, by Professor L. W. Cohen.

Mathematics in European universities, by Professor S. M. Ulam.

A Christmas party was held, and in January fifty new members were admitted at an initiation banquet. A very interesting paper was read at this meeting, entitled

Sir Isaac Newton, by Professor R. E. Langer.

The winners of the competitive mathematical examination sponsored by the chapter were Warren Young, first prize, and David Herwitz, second. Officers for the year were: President, Arne V. Larson; Vice-President, Anne Braun; Treasurer, Henry Rogers; Secretary, Betty Lohr; Faculty Adviser, Professor J. F. Kenney.

Kappa Mu Epsilon, Upsala College

Nine meetings were held throughout the year. At seven of these meetings papers and reports were given by the members and at two meetings the chapter had outside speakers. The topics discussed by the members were:

Why teach mathematics? by Phyllis Gustafson, the president.

A proof from Weierstrass, by Joseph Prieto.

The divine proportion in classic art, by Elizabeth Ebel.

Sources of Euclid, by Marjorie Wolfe.

The history of pi, by Zelda Meisel.

Statistical research, by Hirsch Geller.

At the special midyear meeting the chapter heard an instructive lecture with many visual illustrations from the field of nature and art, entitled

Dynamic symmetry, by Professor William O'Brien of Newark, New Jersey.

On June 3 the society had its annual banquet and initiation of new members. The guest speaker was Mr. Michael McGreal, Principal of the Barringer High School, Newark, who discussed certain problems of teaching high school mathematics. Also at this meeting the officers for the year 1943–44 were installed. They are: President Thales, Zelda Meisel; Vice-President Apollonius, Joseph Prieto; Secretary Abel, Elizabeth Ebel; Treasurer Fibonacci, Lillian Meisel; Historian Gauss, Marjorie Wolfe; Corresponding Secretary Descartes, M. A. Nordgaard.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Mathematics Dictionary. By Glenn James and R. C. James. Van Nuys, California, The Digest Press, 1942. 5+259+22 pages. $3.00.

Mathematics Dictionary. Revised edition., 1943, 8+274+46 pages. $3.00.

The *Mathematics Dictionary* is a handy reference book of definitions, formulas, and tables, covering mathematical topics in arithmetic, algebra, Euclidean and projective geometry, plane and spherical trigonometry, mathematics of finance, statistics, analytic geometry and calculus. Definitions are couched in language suited to the mathematical maturity of the reader who might require them. No attempt is made to cover mathematical terms which arise in advanced mathematical studies beyond the college level, although a few of these words are included. Related concepts are defined together, and illustrative examples are given which are very helpful in clarifying the meaning of technical terms. Many of the definitions in analytic geometry are illustrated by figures which, with one or two exceptions (*e.g.*, the cubic curve $y=kx^3$), are accurately and clearly drawn. The appendix includes tables of logarithms, trigonometric functions, compound interest and annuities, mortality, squares and cubes (and a partial list of square roots and cube roots), denominate numbers, differential and integral formulas, and a list of mathematical symbols.

The revised edition shows a marked improvement over the first edition, because of its clearer typography, because of its more adequate tables in the appendix, because of many improved definitions and illustrative examples, and because of the elimination of many errors. In the first edition the reviewer noted over a hundred errors, varying from misprints to inadequate or faulty definitions. About half of these have been corrected in the revised edition.

Of the errors in the revised edition, the following appear to be the most serious.

(1) The word *harmonic*, which has so many connotations in mathematics, seems to be unfortunate in the treatment it has received. Although La Place's equation is defined, no mention is made of the fact that its solutions are called *harmonic functions*. Instead, a *harmonic function* is said to be the same as a *compound harmonic function* and is defined to be a trigonometric polynomial. Furthermore, *harmonic conjugates with respect to two points* are defined to be "*the two points which divide the line through the two points internally and externally in the same ratio*", rather than "any two points which divide . . . ."

(2) *Stirling's formula, or series*, is defined to be the *same as Maclaurin's*

series. No mention is made of the important approximation $n! \sim (2\pi n)^{1/2}(n/e)^n$, which is commonly called *Stirling's formula*.

(3) The definitions given for certain terms used in navigation, such as *knot*, *middle latitude sailing*, *triangle of plane sailing* are questionable. Although certain text book writers may perhaps be found who would refer to *knots per hour*, the reviewer feels that a knot is a unit of speed, and should never be defined to be "the same as a nautical mile," as is done in this *Dictionary*. In discussing the *triangle of plane sailing* it should be the *departure*, rather than the *difference in longitude*, which is taken as the east-west side of the right triangle. *Middle latitude sailing* does not consist in "sailing on the middle latitude of two places," as the *Dictionary* avers, but in computing the departure along the middle latitude parallel between the meridians of the two places.

(4) *Unit density* is defined to be "the density of some volumetric unit; same as the density of the entire body when the density is constant" . . . rather than "the density of a substance whose mass is numerically equal to its volume."

(5) The tractrix is defined to be: "*the evolute of the catenary; a curve the lengths of whose tangents are equal.*" However, the evolute of the catenary does not have this stated property, and is not called a tractrix, so far as the reviewer is aware.

(6) In several places throughout the *Dictionary*, reference is made to the inverse trigonometric functions, but in most of these places no mention is made of radian measure except in referring to the graphs. However, under the topic: *Value—principal value of an inverse trigonometric function*, we find the statement that "the principal value of the arc-sine is between $-\pi/2$ and $\pi/2$," and then "*e.g.*, the principal value of $\sin^{-1}(1/2)$ is 30°." The reviewer feels that radian measure should be used throughout in the principal definitions of these functions, and that $\sin^{-1}(1/2) = \pi/6$.

Certain matters on which there may be a difference of opinion seem worthy of mention.

(7) Although under *circular*, a *circular cylinder* is defined to be a cylinder whose right section is a circle, a *circular cone* is defined to be a cone whose base is a circle. Granted that the latter definition was used by the ancients, certain modern authors prefer to define a circular cone to be a cone of revolution, and the reviewer feels that this definition should also be mentioned, if only to be consistent with the definition given for the circular cylinder.

(8) The *matrix* seems to receive inadequate treatment as compared with the *determinant*. Multiplication of matrices is a more fundamental concept than multiplication of determinants, and skew-symmetric matrices than skew-symmetric determinants, yet multiplication and skew-symmetric are here defined for determinants and not for matrices. The reviewer feels, even though many authors of texts may disagree, that a square array of elements in rows and columns should be called a *matrix* and not a *determinant*, and that the determinant should be defined to be a certain polynomial formed from the elements of a square matrix, rather than the "symbolic representation" thereof.

(9) The reviewer agrees with the editors of the *Dictionary* that the right-handed coordinate system in space is preferable to the left-handed system, but feels that in defining the trirectangular trihedral it should not be stated that the determinant of the three sets of direction cosines is $+1$ when the trihedral is *right-handed*, but rather when it *has the same orientation as the axes*. The same criticism applies to the definition of the *vector product* of two vectors.

(10) In the definitions of *inequality* and of *greater* and *less*, the expressions *positive number not zero* occur. It would seem that *not zero* is superfluous in referring to positive numbers.

(11) In defining a *power of a number*, we read "*power is sometimes used in the same sense as exponent.*" The reviewer agrees that the word is sometimes used carelessly in this sense, but feels that the *Dictionary* should not sanction this usage, nor use it in the definition, under *laws of integral exponents*, where it says: "*to raise to a power multiply the exponent by the power*".

(12) Although a good table of Denominate Numbers is given in the Appendix, there is no way by which the reader can find in this *Dictionary* the relationship between avoirdupois and troy weights, each of which uses the grain, the ounce, and the pound, but obviously not all with the same meaning in both systems. It would be of interest, perhaps, to point out that a pound of feathers weighs more than a pound of gold, but an ounce of gold weighs more than an ounce of feathers, since avoirdupois weight is used for feathers, but troy weight is used for gold.

Attention might be called to some misprints and errors found in the revised edition. (Numbers refer to pages.)

(22) Bessel functions: the exponent $x(t-1/t)$ should read $x(t-1/t)/2$.

(58) Cubical: the cubic graph $y=kx^3$ should be tangent to the x-axis.

(77) Significant digits: the last digit to the right is not necessarily significant when it is a 0 preceding the decimal point.

(85) Double root: $(x-r)$ should read $(x-r)^2$.

(95) Irrational equation: $\sqrt{x+1}=\sqrt{x+2}$ is a poor illustration, since it has no roots.

(103) Fermat's theorem: read a^{p-1} in the formula $a^{n-1}\equiv 1 \pmod{p}$.

(105) Half-angle formulas: the formula $\tan \frac{1}{2}A=(1-\cos A)/\sin A$ should be included.

(116) Gravitational constant: the letter g is more commonly used to mean the acceleration of gravity than to mean the constant in the equation $F=gm_1m_2/r^2$, as here defined.

(118) Half-angle formulas of spherical trigonometry: in the given formula $\tan \frac{1}{2}\alpha=r/\sin(s-a)$, it would be preferable to use $\tan r$ instead of r, so that r would then be the inradius, as it is in the analogous plane formulas. Similarly, if R were replaced by $\tan R$ in the half-side formulas, the new R would be the circumradius and would have geometric significance.

(119) Simple hexagon: co-linear should read collinear, to be consistent with the spelling elsewhere.

(130) N-scribed: should read inscribed.

(138) Closed interval: Usually denoted by $[a, b]$ rather than (a, b). The latter refers to an open interval.

(142) Kepler's laws: *area* should read *areas*.

(148) Limaçon: read a instead of $2a$ in second line.

(148) Limit: read $0<|x-a|<\delta$ instead of $|x-a|<\delta$.

(157) Mathematical induction: last line of definition should read "true for all positive integral values of n."

(158) Maximum: the slope may change sign without being zero or infinite if the function has a discontinuous derivative.

(158) Maximum, and (166) Multiple: instead of "first derivative which is not zero" or "first derivative of which it is not a root," read "derivative of lowest order which is . . ."

(160) Mean value theorem or the law of the mean: continuity of derivative should be explicitly required.

(168) Napier's analogies: Letters α, β, γ are used here for angles, whereas A, B, C are used in Gauss' formulas (112). Uniformity of notation would be preferable.

(188) Planimeter: Not synonymous with polar planimeter, since there are planimeters of other types.

(199) Probability curve: exponent $-h^2(y-A)^2$ should read $-h^2(x-A)^2$.

(209) Ratio: the six values of the cross ratio of four points are not all distinct when the values are complex cube roots of unity.

(210) Rationalize: the given example $\sqrt{x-1}=x+2$ has only an extraneous root, and is therefore an unfortunate choice.

(213) Reduction to normal form: to be consistent with earlier definitions, use ω in place of θ in the equation of the line.

(216) Repeated: "a dice" should read "a die."

(220) Rule: Descarte's should read Descartes'.

(222) Second: not "one three hundred sixtieth part of a degree." (1/3600).

(225) Bounded sequence: last word should be *terms*, not *term*.

(237) Solution of trigonometric equation: poor definition. The equation $\theta=\cos\theta$ cannot be solved according to the method given.

(237) Species: a right angle is sometimes considered to be a third species in addition to acute and obtuse angles.

(245) Surface integral: the factor sec β is omitted in the displayed integral.

(250) Tangent to a circle: read "a straight line," not "the straight line."

(252) Taylor's theorem: Read $|y-b|$ instead of $|x-b|$ in the fourth line of the page.

(261) Trigonometric curves: the cotangent is omitted by mistake.

(265) Uniform scale: reference to logarithmic paper may be misleading.

(269) Vector components: the word parallelepiped is misspelled.

(273) Zero of a function: the sign $+$ is used for $=$ in two places.

APPENDIX:

(30) 6,085 feet is not a nautical mile.

(31) 69 miles is not exactly one degree of latitude.

Write 1 U. S. gallon = 231 cu. in., since the imperial gallon is different.

Include "equinox year" and "nickel" among definitions.

(43) The three cube roots of unity are often written 1, ω, ω^2.

Read $\bar{z}$ for z.

Read "modulo p" instead of "modulus p."

(44) $\sinh^{-1}x$ is not an angle.

The reviewer would like to see the following words or phrases listed in the *Dictionary* if and when another revised edition is printed:

Average rate, characteristic equation (of a matrix), *coset* (of a subgroup) *dummy index, equation of time, quire.*

The misprints and errors which have been mentioned above are regrettable but they do not alter the fact that the *Mathematics Dictionary* is a useful reference book for terms used in high school or college mathematics. The *Dictionary* does not pretend to serve the needs of the mathematician engaged in graduate work or research. The editors are to be congratulated for having taken the first difficult steps towards a standardization of mathematical terminology.

J. S. FRAME

A First Course in Mathematics for Students of Engineering and the Physical Sciences. By Edward Baker. New York, D. Van Nostrand Company, 1942. 13+295 pages.

This text presents the topics ordinarily studied in a freshman course of "general mathematics" as offered by most colleges. The treatment, however, is not the purely conventional one; an effort is made, by a shift of emphasis, to meet the needs of students intending to specialize in engineering or in the physical sciences.

Simple vector notions in the plane are explained at the start, and are later used, to a limited extent, in deriving formulas of trigonometry and of plane analytic geometry. The importance of the formulas of analytic trigonometry in transforming and simplifying expressions and equations, especially those arising in the sciences, is very neatly and commendably emphasized; but more exercises might well have been provided to illustrate such transformations. The sections dealing with triangulation are merely adequate, and not, as is often the case, overexpanded; quite enough is presented to meet all practical requirements. The analytic geometry of the plane receives somewhat less attention, in certain respects, than is usually accorded to it in over-all survey courses normally taken by freshmen not majoring in mathematics or the sciences. One wonders why space is provided for even a cursory analysis of the general equation of a conic section, rectangular Cartesian coordinates, in a text which does not discuss tangent lines to any plane curve at all—not even in the too brief and sketchy penultimate chapter devoted to the most rudimentary ideas and operations of

the calculus. In the reviewer's opinion, the determining of plane curves which represent geometric loci, and the sketching of such curves from their equations, should not have been relegated to the last of the chapters dealing with plane analytic geometry, and should have been handled much more expansively. The author appears to regard as unimportant, for first-year science students, the eccentricity and the directrices of the ellipse and the hyperbola; and the reviewer is inclined to concur in this judgment.

The problems listed for solution constitute an outstanding and very attractive feature of the book. They are drawn, in large proportion, from various fields of physical and engineering science, and illuminate in a highly stimulating manner the practical significance and scope of even a first course in college mathematics. Numerous problems were evidently selected with the definite object of developing greater dexterity in arithmetic and algebraic manipulations than most of our non-engineering freshmen attain, or are capable of attaining in their abbreviated courses, in these days of "progressive" hop-skip-jump-and-skid elementary and secondary training. Consider, for illustration, the following problem, which is not at all among the most interesting proposed but would certainly have paralyzed most of the freshmen whom this reviewer has entertained in the classroom in recent years: "An empirical formula for the drop in temperature of a gas passing through a flue is

$$ln\ ln\left(\frac{T_1}{t_1}\right) - ln\ ln\left(\frac{T_2}{t_1}\right) = ML.$$

Show that

$$T_1 = t_1\left(\frac{T_2}{t_1}\right)^{e^{ML}}.$$

Calculate T_1 for $t_1 = 1000$ degrees absolute, $T_2 = 800$ degrees absolute, and $ML = 1.5$."

An instructor not versed in engineering lore would probably appreciate amplified information concerning the physical interpretations of all constants and variables occurring in problems of this type. In the problem quoted, he might guess the meanings of the mysterious M and L; but in many of the other problems stated such guessing would be futile.

The typography and the set-up of pages are attractive, and make this text exceptionally readable. A first edition may be expected to show misprints, but the only really unfortunate and confusing one which the reviewer has observed is found in the last equation on page 158. The instructor will undoubtedly have to come to the student's rescue at that spot. He may also be called upon to supplement the argument there presented by explaining an elementary point of number theory which is assumed without comment. In the statement of the Remainder Theorem for polynomials, as given on page 156, the word "numerically" is misleading and should be deleted. It is inadvertently asserted, on page

226, that "any equation of the second degree (in two variables) represents an ellipse, a hyperbola, or a parabola." This assertion, and that made in the last sentence of the same paragraph, will of course receive, in a second edition, the necessary reupholstering and repair.

The omissions noted, and one or two others, are very satisfactorily offset by several unusual inclusions which are of direct interest to the prospective student of science. Moreover, the author's style is fluent, clear, and pleasing. This book may be recommended for use with classes in which smooth and rapid progress is sought—with no "bogging down" anywhere. It is really a good contribution to the current war effort in education. L. S. HILL

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 596. *Proposed by W. C. Rufus, Observatory of the University of Michigan*

A square phalanx of less than five million soldiers was reinforced by ten equal squares and then formed a square of over ten million which could be divided into 400 equal squares. The reinforced army was less than three times the original. How many soldiers were in each of the 400 squares?

E 597. *Proposed by V. Thébault, San Sebastián, Spain*

Let P be any point in the plane of a triangle ABC. Let (U), (V), (W) denote the circles BCP, CAP, ABP, and (U'), (V'), (W') their images by reflection in the respective sides BC, CA, AB. Also let u, u' be the powers of A with respect to (U), (U'), and let v, v', w, w' be defined analogously. Show that the circles (U'), (V'), (W') are concurrent, and that

$$u + u' + v + v' + w + w' = a^2 + b^2 + c^2.$$

E 598. *Proposed by H. S. Wall, Northwestern University*

Let $g_1, g_2, g_3, \cdots$ be any numbers such that

$$0 < g_p < 1, \qquad (1 - g_p)g_{p+1} > \tfrac{1}{4} \qquad (p = 1, 2, 3, \cdots).$$

Prove that

$$\lim_{p\to\infty} g_p = \tfrac{1}{2}.$$

E 599. *Proposed by U. P. Davis, University of Florida*

In how many different ways can $3n$ objects be arranged in a circle, if there are n different kinds, 3 of each kind?

E 600. *Proposed by J. H. Butchart, Grinnell College*

If the radii of the fixed and rolling circles are a and b respectively, the length of one arch of an epicycloid is $8(a+b)b/a$, and the area bounded by one arch and the fixed circle is

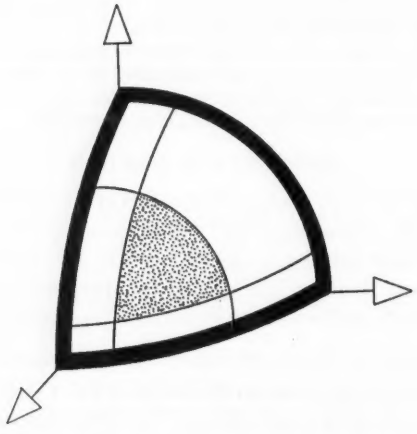
$$\pi(3a^2 + 8ab + 4b^2)b^2/a(a + 2b).$$

Corresponding formulas for the hypocycloid are obtained by changing the sign of b. Prove these results synthetically.

SOLUTIONS

A Triangle of Small-circular Arcs

E 559 [1943, 120]. *Proposed by M. A. Sadowsky, Illinois Institute of Technology*



Let a, b, c be non-negative numbers satisfying $a^2+b^2+c^2<1$. Prove that the three planes $x=a$, $y=b$, $z=c$ will cut out from the first octant of the unit sphere $x^2+y^2+z^2=1$ a curvilinear triangle of area

$$\begin{aligned}\tfrac{1}{2}\pi(1 - a - b - c) &+ a(\text{arc sin } b_a + \text{arc sin } c_a)\\ &+ b(\text{arc sin } c_b + \text{arc sin } a_b)\\ &+ c(\text{arc sin } a_c + \text{arc sin } b_c)\\ &- \text{arc sin } b_c c_b - \text{arc sin } c_a a_c - \text{arc sin } a_b b_a),\end{aligned}$$

where a_b stands for $a/\sqrt{1-b^2}$, and so on.

Solution by E. P. Starke, Rutgers University. The standard method for espressing the area of the given curvilinear triangle as a double integral gives

$$\int_a^{\sqrt{1-b^2-c^2}} \int_b^{\sqrt{1-c^2-x^2}} \frac{dydx}{\sqrt{1-x^2-y^2}}.$$

Performing the first integration, we have

$$A = \int_a^{\sqrt{1-b^2-c^2}} \left(\arccos \frac{c}{\sqrt{1-x^2}} - \arcsin \frac{b}{\sqrt{1-x^2}} \right) dx. \tag{1}$$

The final integration is effected by use of the formula

$$\int \arccos \frac{c}{\sqrt{1-x^2}}\,dx = x \arccos \frac{c}{\sqrt{1-x^2}} - c \arcsin \frac{x}{\sqrt{1-c^2}} + \arcsin \frac{cx}{\sqrt{1-c^2}\sqrt{1-x^2}}$$

and an analogous formula for the second term of (1). Using these and substituting the limits in (1), we find, after rearranging terms,

$$\begin{aligned} A = {} & \arcsin \frac{c\sqrt{1-b^2-c^2}}{\sqrt{b^2+c^2}\sqrt{1-c^2}} + \arcsin \frac{b\sqrt{1-b^2-c^2}}{\sqrt{b^2+c^2}\sqrt{1-b^2}} \\ & - c \arcsin \frac{\sqrt{1-b^2-c^2}}{\sqrt{1-c^2}} - b \arcsin \frac{\sqrt{1-b^2-c^2}}{\sqrt{1-b^2}} \\ & - a \arccos c_a + c \arcsin a_c + a \arcsin b_a \\ & + b \arcsin a_b - \arcsin a_c c_a - \arcsin a_b b_a. \end{aligned} \tag{2}$$

If the first two terms of (2) are combined as $(\frac{1}{2}\pi - \arcsin b_c c_b)$, and the next three terms are written as

$$-c(\tfrac{1}{2}\pi - \arcsin b_c) - b(\tfrac{1}{2}\pi - \arcsin c_b) - a(\tfrac{1}{2}\pi - \arcsin c_a),$$

we have the proposed form for the area A.

Also solved by W. E. Büker and Eugene Sherwood (together).

Parallel Simson Lines

E 561 [1943, 200]. *Proposed by Howard Eves, Syracuse University*

Given two triangles inscribed in the same circle and such that the Simson lines with respect to one triangle of the vertices of the other are concurrent (as in E 535), prove that the Simson lines with respect to the two triangles of a point on the common circumcircle are parallel.

Solution by the Proposer. From the lemma of E 535 [1943, 261] we see that the theorem is true for the point A. The theorem then follows for any other point

on the circumcircle by the first corollary of art. 326 in R. A. Johnson's *Modern Geometry*.

A Perfect Square with Nine Different Digits

E 562 [1943, 200]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a number of the form *ab0cd* whose square contains the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9.

Solution by M. L. Constable, Philadelphia. Since the square is less than 987654321, the number lies between 10234 and 31098. Since the sum of the digits is 45, the square is divisible by 9, and the number by 3. Therefore, of the 126 possible *combinations* of these nine digits taken four at a time. we select the 42 whose digit-sums are divisible by 3. We form the *permutations* of these combinations, reject those greater than 3198, and insert a zero in the third place of each one remaining. Now we note that, because of the zero in *ab0cd*, the last three digits of its square will be found in the last three digits of the first hundred squares. Therefore a table will show that *cd* cannot be any of

12, 15, 21, 26, 32, 35, 38, 45, 46, 47, 48, 49, 51, 52, 53,
62, 63, 64, 65, 71, 76, 78, 83, 84, 85, 92, 95, 97, 98.

We square the remaining numbers, and find that the squares of

18072, 19023, 23019, 29034

are respectively 326597184, 361874529, 529874361, 842973156.

Also solved by W. E. Buker, C. W. Emmons, Alexander Russ, E. P. Starke, Alan Wayne, and the proposer. Wayne points out that the second and third solutions show an interesting "inversion" in the digits of the numbers and of the squares.

A Hyperbolic Group of Lines

E 563 [1943, 200]. *Proposed by N. A. Court, University of Oklahoma*

Let A', B', C', D' be the antipodes of the circumcenter O of a tetrahedron $ABCD$ on the respective spheres $OBCD$, $OCDA$, $ODAB$, $OABC$. Show that the lines AA', BB', CC', DD' are generators of a quadric. May this quadric be a cone?

Solution by the Proposer. The antipodes A' of O on the sphere $OBCD$ is the point common to the three planes perpendicular to the lines OB, OC, OD at the respective points B, C, D. But these three planes are the tangent planes at B, C, D to the circumsphere of $ABCD$; hence A' is a vertex of the tangential tetrahedron of $ABCD$. Thus the line AA' and the three analogous lines BB', CC', DD' join the vertices of $ABCD$ to the corresponding vertices of its tangential tetrahedron. The proposition now follows from art. 322 of the proposer's *Modern Pure Solid Geometry*.

The four lines will be concurrent if the tetrahedron $ABCD$ is isodynamic (*ibid.*, pp. 280, 281).

A Question of Divisibility

E 564 [1943, 200]. *Proposed by Ivan Niven, Purdue University*

Let a, b, and n be any positive integers such that n divides a^n-b^n. Prove that n divides $(a^n-b^n)/(a-b)$.

Solution by the Proposer. Denote $a-b$ by c. Let p^m be the highest power of a prime p which divides n. If p does not divide c, we can ignore it. If p divides c, we write

$$\frac{a^n-b^n}{c}=\frac{(b+c)^n-b^n}{c}=\sum_{i=1}^{n}\binom{n}{i}b^{n-i}c^{i-1},$$

and show that p^m divides each term of this sum. Since $\binom{n}{i}$ is a multiple of n divided by $i!$, we need prove merely that p divides c^{i-1} to as high a power as it divides $i!$. Now the number of times p divides $i!$ is evidently

$$\sum_{j=1}^{\infty}\left[\frac{i}{p^j}\right]<\sum_{j=1}^{\infty}\frac{i}{p^j}=\frac{i}{p-1}\leqq i.$$

Noting the strict inequality in the first step, we find that our proposition is proved, because p^{i-1} divides c^{i-1}, by hypothesis.

Also solved by E. P. Starke and (incompletely) by R. K. Allen, D. H. Browne, Thor Eriksson, and Joseph Rosenbaum.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4101. *Proposed by R. C. Buck, Harvard University*

Show that

$$D(n)=\begin{vmatrix}(1,1)^\lambda, & (1,2)^\lambda, & \cdots, & (1,n)^\lambda\\ (2,1)^\lambda, & (2,2)^\lambda, & \cdots, & (2,n)^\lambda\\ \cdot & \cdot & & \cdot\\ \cdot & \cdot & & \cdot\\ (n,1)^\lambda, & (n,2)^\lambda, & \cdots, & (n,n)^\lambda\end{vmatrix}$$

$$=(n!)^\lambda\left(1-\frac{1}{2^\lambda}\right)^{[n/2]}\left(1-\frac{1}{3^\lambda}\right)^{[n/3]}\left(1-\frac{1}{5^\lambda}\right)^{[n/5]}\cdots,$$

where (i,j) means the greatest common divisor of the integers i, j.

4102. *Proposed by Hüseyin Demir, Columbia University*

Let O and I be respectively the circumcenter and incenter of a given triangle ABC. Let A_0, B_0, C_0 be points taken respectively on BC, CA, AB so that the sums of the algebraic distances of each point to two other sides are equal to a given length l. Prove synthetically that: (1) The points A_0, B_0, C_0 are collinear; (2) The sum of distances to the sides of ABC of points on $A_0B_0C_0$ is the constant l; (3) the line $A_0B_0C_0$ is perpendicular to the line OI.

4103. *Proposed by V. Thébault, San Sebastián, Spain*

In the system of base $n+1$ the product $P=N\cdot L$ is formed where the number N of $n-1$ digits in descending order is $n(n-2)(n-3)\cdots 321$, and L is less than n and prime to n. If we have $L<n/2$, then the product P has n distinct digits chosen in suitable order from $0, 1, 2, \cdots, n$, and the missing digit is $n-L$. If $L>n/2$, the digit $n-1$ appears twice in P, and the missing digits are n and $(n-1-L)$, the remaining digits being distinct.

Dedicated to E. P. Starke.

SOLUTIONS

Perfect Squares

4048 [1942, 479]. *Proposed by V. Thébault, San Sebastián, Spain*

Find a number of six digits $N=abcdef$ such that the product NN' is a perfect square, where $N'=defabc$.

Solution by Daniel Finkel, Mt. Rainier, Md. Let x and y be three digit numbers such that $(10^3x+y)(10^3y+x)=z^2$ where z is an integer. Solving for x we have

$$2000x = -(10^6+1)y + [(10^6-1)^2y^2 + 4000z^2]^{1/2}. \tag{1}$$

In order to make x rational, set

$$[az + (10^6-1)y]^2 = 4000z^2 + (10^6-1)^2y^2;$$

and we then find that

$$z = \frac{2a(10^6-1)y}{4000-a^2}, \qquad x = \frac{az-2y}{2000}, \qquad 0 < a \leqq 63, \tag{2}$$

$$10^6 - 1 = 27\cdot 37\cdot 7\cdot 11\cdot 13.$$

We now investigate the conditions under which $4000-a^2$, with a an integer, may have for a factor one of 7, 11, 13, 37. Since $4000\equiv 3 \bmod 7$ and 3 is not a quadratic residue mod 7, the number 7 cannot be a factor for any value of a; and similarly 11 cannot be a factor. We then find that 13, 37 will be factors respectively for

$$a \equiv \pm 3 \bmod 13; \qquad a \equiv \pm 2 \bmod 37.$$

Hence the possible values for a are

$$a = 2, 3, 10, 16, 23, 29, 35, 36, 39, 42, 49, 55, 62.$$

Upon trial a solution is found for $a=16$ giving $z=8547y$, $x=547y/8$; and then setting $y=8$ we have the solution

$$N = 547008, \qquad N' = 8547, \qquad z = 68376.$$

Editorial Note The trivial solution $x=y$ is given by $a=2$. For $a=3$ we get $x=692$, $y=307$, $z=461538$, which is the solution given by the proposer without indication of his method of deduction. For $a=10$ we find that $x=641$, $y=25$, $z=128205$.

Identities in Cotangents

4050 [1942, 549]. *Proposed by Arnold Dresden, Swarthmore College*

If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that no two differ by a multiple of π, prove that

$$\sum_{k=1}^{n} \prod_{i=1, i\neq k}^{n} \cot (a_k - a_i) = \sin \frac{n\pi}{2}.$$

Solution by the Proposer. For $n=2$ the formula is true, since $\cot (a_1-a_2) + \cot (a_2-a_1) = 0 = \sin \pi$. Suppose that it is true for $n=2, 3, \cdots, k$; we shall prove that then it is true also for $n=k+1$. For we have then

$$(1) \qquad \sum_{j=1}^{k} \cot (a_1 - a_{k+1}) \prod_{i=1, i\neq j}^{k} \cot (a_j - a_i) = \sin \frac{k\pi}{2} \cot (a_1 - a_{k+1}),$$

$$(2) \qquad \prod_{i=2}^{k+1} \cot (a_1 - a_i) + \sum_{j=2}^{k} \cot (a_j - a_1) \cot (a_1 - a_{k+1}) \prod_{i=2, i\neq j}^{k} \cot (a_j - a_i) = \sin \frac{k\pi}{2} \cot (a_1 - a_{k+1}).$$

Since $\cot P \cot Q = 1 + \cot (P+Q)[\cot P + \cot Q]$, the second term of (2) becomes after setting $P=a_j-a_1$, $Q=a_1-a_{k+1}$,

$$(3) \qquad \sum_{j=2}^{k} \prod_{i=2, i\neq j}^{k} \cot (a_j - a_i) + \sum_{j=2}^{k} \prod_{i=1, i\neq j}^{k+1} \cot (a_j - a_i) + \cot (a_1 - a_{k+1}) \sum_{j=2}^{k} \prod_{i=2, i\neq j}^{k+1} \cot (a_j - a_i).$$

The first term of (3) is equal to $\sin (k-1)\pi/2$, and the third term is equal to

$$\cot (a_1 - a_{k+1}) \left[\sin \frac{k\pi}{2} - \prod_{i=2}^{k} \cot (a_{k+1} - a_i) \right].$$

Hence we have after collection of terms

$$\sum_{j=1}^{k} \prod_{i=1, i\neq j}^{k+1} \cot(a_j - a_i) + \sin\frac{(k-1)\pi}{2}$$

$$+ \cot(a_1 - a_{k+1})\left[-\sin\frac{k\pi}{2} + \sin\frac{k\pi}{2} - \prod_{i=2}^{k} \cot(a_{k+1} - a_i)\right];$$

$$\sum_{j=1}^{k+1} \prod_{i=1, i\neq j}^{k+1} \cot(a_j - a_i) = -\sin\frac{(k-1)\pi}{2} = \sin\frac{(k+1)\pi}{2};$$

and the induction proof is complete.

Identities in Cotangents

4051 [1942, 549]. *Proposed by Arnold Dresden, Swarthmore College*

If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that none of them and none of the differences is a multiple of π, show that

$$\sum_{j=1}^{n} \cot a_j \prod_{i=1, i\neq j}^{n} \cot(a_j - a_i) + (-1)^n \prod_{i=1}^{n} \cot a_i = \sin\frac{(n+1)\pi}{2}.$$

Solution by the Proposer. Since $\cot(a_1-a_2)[\cot a_2-\cot a_1]=1+\cot a_1 \cot a_2$, the formula is true for $n=2$. Suppose that the formula holds for $n=2, 3, \cdots, k$, then we have from the formula in 4050

$$\sum_{j=1}^{k+1} \cot a_1 \prod_{i=1, i\neq j}^{k+1} \cot(a_j - a_i) = \sin\frac{(k+1)\pi}{2} \cot a_1, \tag{1}$$

$$\cot a_1 \prod_{i=2}^{k+1} \cot(a_1 - a_i) + \sum_{j=2}^{k+1} \cot a_1 \cot(a_j - a_1) \prod_{i=2, i\neq j}^{k+1} \cot(a_j - a_i)$$

$$= \sin\frac{(k+1)\pi}{2} \cot a_1. \tag{2}$$

Since $\cot(a_j-a_1)\cot a_1=1+\cot a_j\,[\cot(a_j-a_1)+\cot a_1]$, the second term of (2) becomes

$$\sum_{j=2}^{k+1} \prod_{i=2, i\neq j}^{k+1} \cot(a_j - a_i) + \sum_{j=2}^{k+1} \cot a_j \prod_{i=1, i\neq j}^{k+1} \cot(a_j - a_i)$$

$$+ \cot a_1 \sum_{j=2}^{k+1} \cot a_j \prod_{i=2, i\neq j}^{k+1} \cot(a_j - a_i). \tag{3}$$

The first term of (3) is equal to $\sin k\pi/2$ from the formula of 4050, and by the induction assumption the third term is equal to

$$\cot a_1\left[\sin\frac{(k+1)\pi}{2} - (-1)^k \prod_{i=2}^{k+1} \cot a_i\right].$$

Hence we have after collection of parts and a cancellation

$$\sum_{j=1}^{k+1} \cot a_j \prod_{i=1, i\neq j}^{k+1} \cot (a_j - a_i) + (-1)^{k+1} \prod_{i=1}^{k+1} \cot a_i = -\sin \frac{k\pi}{2} = \sin \frac{(k+2)\pi}{2},$$

and the induction proof is complete.

Ellipse and Extremal Triangles

4053 [1942, 549]. *Proposed by E. P. Starke, Rutgers University*

Show that all triangles inscribed in an ellipse and having their centroids at the center of the ellipse have the same area, which is the greatest possible area for an inscribed triangle.

Show that all triangles circumscribed about an ellipse and having their centroids at the center of the ellipse have the same area, which is the least possible area for a circumscribed triangle.

Solution by P. C. Hammer, Oregon State College. To solve the two given problems we shall prove a more general theorem.

THEOREM *Let* $(a \cos \theta_i, b \sin \theta_i)$, $i=1, 2, \cdots, n \geqq 3$, *be the cartesian coordinates of the vertices of a convex polygon inscribed in the ellipse* $x=a \cos \theta$, $y=b \sin \theta$, *and also the coordinates of the points of tangency of a circumscribed polygon, where* $0<\theta_1<\theta_2< \cdots <\theta_n \leqq 2\pi$. *The necessary and sufficient conditions that the inscribed polygon have a maximum and that the circumscribed polygon have a minimum area are*

$$\theta_2 - \theta_1 = \theta_3 - \theta_2 = \cdots = \theta_n - \theta_{n-1} = 2\pi + \theta_1 - \theta_n = 2\pi/n. \tag{1}$$

Proof. The transformation $x=ax'$, $y=by'$ takes the ellipse into the unit circle, straight into straight lines, and changes all areas by the constant factor $1/ab$. Hence the problem is reduced to one of maximizing the areas of polygons inscribed in the unit circle and of minimizing the areas of circumscribed polygons. The transformed inscribed polygon has the vertices $(\cos \theta_i, \sin \theta_i)$ and its area K_1 is given by

$$2K_1 = \sin (\theta_2 - \theta_1) + \cdots + \sin (\theta_n - \theta_{n-1}) + \sin (\theta_1 - \theta_n). \tag{2}$$

Now we may place $\theta_n = 2\pi$ due to the symmetry of the circle, and using partial derivatives, we find readily that conditions (1) are necessary and sufficient to maximize K_1. The area of the corresponding polygon inscribed in the ellipse is

$$\tfrac{1}{2}nab \sin \frac{2\pi}{n}.$$

The area of a polygon circumscribed about the unit circle is

$$K_2 = \cot \phi_1 + \cot \phi_2 + \cdots + \cot \phi_n,$$

where the ϕ_i's are halves of the vertex angles of the polygon. The minimum of K_2 is readily obtained using partial derivatives and leads to $\phi_1=\phi_2= \cdots =\phi_n$. The

maximal polygon is thus regular and conditions (1) are fulfilled. The area of the corresponding circumscribed polygon containing the ellipse is

$$nab \tan \frac{\pi}{n}.$$

To apply the theorem to the specific problems we observe that, since all regular polygons have centroids at their centers and as (0, 0) is fixed under the transformation, it is necessary for the extremizing polygons to have centroids at the center of the ellipse. The converse, however, is true only if $n=3$. If the centroid of an inscribed triangle is at (0, 0) then we have the centroid of the transformed triangle is at (0, 0). That such a triangle is equilateral is an immediate consequence of the fact that, if a median of a triangle passes through the circumcenter, the triangle is isosceles. Likewise, if a triangle circumscribed about a circle has its centroid at the center of the circle, it is equilateral due to the fact that, if a median and a vertex angle bisector in a triangle coincide, the triangle is isosceles. From the theorem we then have that in case $n=3$ coincidence of centroid with the center of the ellipse is necessary and sufficient to extremize the areas.

Solved also by J. A. Bullard, H. S. M. Coxeter, H. Eves, H. Scheffé, and the proposer.

Editorial Note. The remaining solvers considered only triangles; Bullard and the proposer made no use of the projection theorem and used partial derivatives in the consideration of the extremal properties, and the proposer referred to the solution of E 191 [1936, 433]. The other solvers used the projection theorem and considered as known the extremal properties for the circle and triangle. The theorems that the regular polygons have respectively the maximum and minimum areas of all convex polygons of the same number of sides inscribed and properly circumscribed about the unit circle is a consequence of the following elementary theorem.

Given the real, continuous, and finite function $f(x)$ of the real variable x which is everywhere concave, or convex, toward the x axis for an interval I for x; then, for an arbitrarily chosen set of n values x_i of I, not all equal, we have

$$\sum_{i=1}^{n} f(x_i) \lesseqgtr nf\left[\sum_{i=1}^{n} x_i/n\right],$$

according as $f(x)$ is always concave or convex toward the x axis.

The proof is obvious. The m points $P_1, P_2, \cdots, P_m$ on the curve $y=f(x)$ are the vertices of a convex polygon P, and the center of mass of the positive integral masses $p_1, p_2, \cdots, p_m$ at these vertices, whose sum is n, lies within this polygon, or the chord if $m=2$. For the polygon of n sides inscribed in the unit circle we need to consider only such as have the center of the circle in their interior. For, if the inscribed polygon $A_1A_2 \cdots A_n$ does not have the center in its interior, there is a vertex, say A_1, such that no vertex lies within say the lower

semicircumference arc for the diameter $A_1'OA_1$. Draw the diameter $A_{n-1}OA_{n-1}'$ and take the point B_n within the arc $A_1'A_{n-1}'$ which continues to A_1. Then, since $n \geqq 3$, arc $A_{n-1}A_n \leqq$ arc $A_{n-1}'A_1 <$ arc B_nA_1, and chord $A_{n-1}A_n <$ chord B_nA_1. The polygon $A_1A_2 \cdots A_{n-1}B_n$ has an area which exceeds that of the original polygon by the excess of the area of triangle KB_nA_1 over that of the similar triangle $KA_{n-1}A_n$, where K is the intersection of $A_{n-1}B_n$ and A_nA_1. The second polygon has the center O in its interior. Now let x_i be the angle at the center subtended by the ith side of the inscribed polygon of n sides with O in its interior where $0 < x_i < \pi$, and the sum of the x_i's is 2π. We have to consider the maximum of the sum of the sines of these n angles. By the above theorem the area of such a polygon if its angles are not all equal is less than the area of the regular polygon. For the properly circumscribed polygon, *i.e.*, containing the center in its interior, we take x_i as half the angle at the center subtended by the ith chord of contact so that $0 < x_i < \pi/2$ and the sum of the x_i's is π. The proof follows in a similar manner by considering the sum of the tangents of the angles x_i.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Associate Professor H. F. Archibald of Keuka College, Keuka Park, New York, has been promoted to a professorship.

Professor W. C. Brenke of the University of Nebraska has retired from the chairmanship of the department of mathematics but will continue his other duties during the current academic year.

Assistant Professors C. T. Male and A. H. Fox of Union College have been promoted to associate professorships.

Assistant Professor C. E. Clark of Bradley Polytechnic Institute has been appointed to an associate professorship at Emory University.

Dr. J. B. Coleman has been appointed to a professorship at the University of Richmond.

Dr. L. M. Garrison of Alabama Polytechnic Institute has been appointed to an assistant professorship at Louisiana Polytechnic Institute.

Dr. R. W. Gibson of the University of Illinois has been appointed to an assistant professorship at Kansas State College.

Assistant Professor E. H. C. Hildebrandt of the State Teachers College, Upper Montclair, New Jersey, has been appointed to an assistant professorship at Northwestern University.

Dr. Lois Kiefer of the University of Illinois has been appointed head of the department of mathematics at State Teachers College, Silver City, New Mexico.

Associate Professor F. H. Miller of the Cooper Union School of Engineering has been promoted to a professorship and has been appointed head of the department of mathematics.

Professor M. G. Moore of Tri-State College, Angola, Indiana, has been appointed to an assistant professorship at Bradley Polytechnic Institute.

Drs. Vladimir Morkovin and W. H. Pell of Brown University have accepted positions as research engineers in aerodynamic design at the Bell Aircraft Corporation, Buffalo.

J. D. Novak of the University of Minnesota has been appointed head of the department of mathematics and physics at MacMurray College, Jacksonville, Illinois.

Dr. D. H. Rock of Brown University has been appointed assistant professor of mathematics at Rhode Island State College.

Dr. Robert Schatten has been appointed to an assistant professorship at the University of Vermont.

Dr. G. E. Schweigert of the University of Missouri has been appointed to an assistant professorship at Purdue University.

Dr. C. J. Thorne of Washington University has been appointed to an assistant professorship at Louisiana State University.

The following appointments to instructorships are announced:
Bard College: Dr. H. B. Mann
Brown University: P. T. Miekle, W. E. Barnes
Cooper Union School of Engineering: S. G. Roth
Indiana University: Dr. Marion D. Wetzel
Purdue University: Dr. Fred Kiokemeister
University of Minnesota: Dr. Louis Garfin, Dr. C. M. Jensen
University of Pennsylvania: R. T. Luginbuhl
University of Rochester: Dr. H. S. Kieval
University of Washington: Dr. J. M. Kingston

Professor Edward Helly of Illinois Institute of Technology died on November 28, 1943.

Mr. L. L. Locke, who retired in 1942, died August 28, 1943. He was a charter member of the Mathematical Association.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

THE NAVY COLLEGE TRAINING PROGRAM

The Navy College Training Program (V-12) was inaugurated upon July 1, 1943, in nearly 200 colleges and universities. This new program superseded the previous V-1 and V-7. Its purpose, as outlined by the Navy Department, is to produce Naval officers chosen from high school seniors, high school graduates, and college students who appear to have the potentialities for ultimate selection as officers after carefully prescribed college training. The plan provides that this college training shall be carried on while the men are on active duty, in uniform, receiving pay, and under general military discipline.

The selection of the educational institutions to participate in both the Army and the Navy Programs is made by a joint committee consisting of representatives of the Armed Forces and the War Manpower Commission. In the event of failure on the part of the members of the committee to agree, the final decision is made by the chairman of the War Manpower Commission. Institutions are chosen not only on the basis of educational facilities, but also after a study of available housing and messing accommodations. To great extent, institutions accepted have been designated as Navy institutions or as Army institutions; that is, only a few institutions have been awarded both Army and Navy programs.

The Navy has insisted that all colleges and universities holding Navy contracts offer three 16-week terms in each calendar year, the first term to begin on or about July 1, 1943, the next term to start approximately November 1, the next one about March 1, and so on. Such a uniform calendar is regarded as essential in view of the fact that the transfer of students from one college to another will frequently be necessary.

Of the nearly 70,000 students assigned to various institutions on July 1, approximately 80 per cent were students already in college who were previously enlisted in class V-1 or V-7 or who held probationary commissions in the U. S. Naval Reserve or were enlisted in the Marine Corps or Coast Guard Reserves. Consequently, colleges holding Navy contracts received transfers at all levels, from second-term freshmen to second-term seniors. None of these transfer students were required to enter the new, fully prescribed V-12 curriculum, but certain minimum requirements, including mathematics, were specified, and, within the limits of a student's major interest, it was hoped that he would wish to include as many of the V-12 subjects as possible.

Freshman students entering the rigid V-12 Program were drawn from two sources. More than 300,000 young men took the first Army-Navy test given last

April 2; many who successfully passed this test and who designated the Navy as their preference started their V-12 studies upon July 1; other students of this group have been assigned to institutions for other semesters. The second test, open to male high school seniors in their last semester and to graduates, who will reach their seventeenth birthday but not their twenty-second birthday by March 1, 1944, was given upon November 9, 1943. Ultimately, most men entering the V-12 Program will come from those who have successfully passed the Army-Navy test. In addition to this group, a limited number of Navy men who have seen duty with the fleet have also been assigned to the V-12 Program; these men are high school graduates, and were chosen upon the basis of their past record with the Navy. Among this latter group are many students who have a deficient background in mathematics and science, and a large number of institutions have found it necessary to set up special courses to accommodate them. It is probable that more attention will be paid in the future to the academic qualifications of this group of assignees.

The Navy has specified that each institution will give examinations in the various V-12 courses according to its own practices. It should be noted, however, that the Navy Department will give a qualifying examination of its own to all first year college students near the end of their second term. The usual scholastic standards of the institution are to prevail, and classes are to be of normal size. It is expected that standard textbooks will be used and that the instructor will select the text, provided the one designated is generally recognized as standard. Each institution will determine whether or not credit toward a degree shall be given for the completion of courses in the various curricula. Inasmuch as the content of these courses is practically equivalent to that of standard college courses in the same subjects, it is hoped that credit will be given quite generally.

The several types of officer candidates in the V-12 Program will complete training as follows: Aviation Candidates, 2 terms; Deck Candidates, and Supply Corps Candidates, 4 terms; Pre-Medical and Pre-Dental Corps Candidates, 5 terms; Engineer Candidates, 6 terms; Engineer Specialist Candidates, including Physics Majors, Civil Engineer Corps Candidates, Construction Corps Candidates, Pre-Chaplain Corps Candidates, Aerology Specialist Candidates, 8 terms. All curricula contain two terms of elementary mathematical analysis, and all technical programs also involve two terms of calculus. Descriptions of the various courses in mathematics follow.

Mathematical Analysis I. 5 lecture-recitation periods per week.

This course is designed for the student with a limited high school background in mathematics. Within the limits of this background, the following areas will be considered. Elementary College Algebra: fundamental concepts, variable, constant, function; review of axioms, elementary operations; factoring; fractions; formulas; the graph; linear equations; simultaneous linear equations in two unknowns; quadratic equations; exponents and radicals; variation. Trigonometry: angles and their measures, trigonometric functions, linear interpola-

tion and use of tables; right triangles; fundamental identities; logarithms, including introduction to use of slide rule; functions of multiple angles, addition formulas; identities; inverse trigonometric functions; trigonometric equations; laws of sines, cosines, and tangents; oblique triangles.

Mathematical Analysis II. 5 lecture-recitation periods per week. Prerequisite: Mathematical Analysis I.

A continuation of Mathematical Analysis I. The following areas will be considered. Trigonometry: introduction to spherical trigonometry. Analytic Geometry: points in rectangular and polar coordinate systems; analytic equivalents of distance, slope, *etc.*; loci; straight line; circle; conic sections; polar and parametric equations; introduction to solid analytic geometry, including cylindrical and spherical coordinate systems. College Algebra: determinants and solution of systems of equations; simultaneous quadratics (with graphical solution); theory of equations; complex numbers.

Mathematical Analysis III. 5 lecture-recitation periods per week. Prerequisite: 2½ or more units of high school algebra.

College Algebra: fundamental concepts, laws, operations; review of factoring, fractions, linear equations, the graph, quadratic equations and simultaneous quadratic equations, binomial theorem; variation; progressions; determinants and systems of linear equations; exponents, radicals; logarithms. Trigonometry: angles and their measures; the trigonometric functions; significant figures and approximate computation; linear interpolations and use of tables; right triangles; identities; functions of multiple angles and addition formulas; inverse functions; trigonometric equations; oblique triangles; right and oblique triangles in spherical trigonometry.

Mathematical Analysis IV. 5 lecture-recitation periods per week. Prerequisite: Mathematical Analysis III.

A continuation of Mathematical Analysis III. Analytic Geometry: points in rectangular and polar coordinate systems; distance, slope, angle between lines; loci; straight line; circle; conic sections; polar and parametric equations; tangents and normals; curve tracing in various systems; translation and rotation; empirical determinations (curve fitting); direction cosines and numbers; the plane and line; quadric surfaces and sections; cylindrical and spherical coordinates. College Algebra: permutations, combinations, and probability; theory of equations; complex numbers and DeMoivre's Theorem.

Calculus I. 4 lecture-recitation periods per week. Prerequisite: first year mathematics.

Functions: limits and limit theorems (without proof); the derivative and its interpretations; derivatives of algebraic functions; maxima and minima; rates; derivatives of transcendental functions; applications including Newton's method of approximation, and tangents and normals; derivatives of higher order; the differential with applications; definite integral with applications such as length, area, surfaces, and volume, moments, centroids, moments of inertia; improper integrals.

An early introduction of the integral calculus into Calculus I is desired so that Analytical Mechanics I may be taught from that background.

Calculus II. 4 lecture-recitation periods per week. For Engineer Specialist Candidates *c* and *d* (Electrical Engineering), Calculus II will be given in 6 lecture-recitation periods per week for 8 weeks, and will be immediately followed by Calculus III meeting a like number of times per week for the last 8 weeks of the term.

Applications of the definite integral such as work and attraction; curvature; curve tracing; indeterminate forms; series of constant terms; power series with Taylor's and Maclaurin's theorems with remainder term, and with applications in integration; partial differentiation with applications; multiple integrals with applications.

In the case where the Calculus is divided into 5-period and 3-period courses, some of the material in Calculus II will be advanced to Calculus I.

Calculus III—Differential Equations. 6 lecture-recitation periods per week. Prerequisite: Calculus II.

To be given in the second term as a continuation of Calculus II. Differential Equations will then be given 6 hours per week for 8 weeks, and will cover ordinary equations of the first order and simple ordinary equations of the second order; singular solutions; linear equations with constant coefficients and applications to the physical sciences, with practice in setting up and solving the differential equations; approximate methods; systems of differential equations; an introduction to partial differential equations.

Navigation and Nautical Astronomy I. 3 lecture-recitation periods per week. Prerequisite: first year mathematics.

Brief review of plane and spherical trigonometry and use of logarithms; instruments; compass corrections; terrestrial factors; the sailings; dead reckoning and transverse sailings; piloting; seamanship; weather.

Navigation and Nautical Astronomy II. 3 lecture-recitation periods per week. Prerequisite: Navigation and Nautical Astronomy I.

Celestial factors; the astronomical triangle; altitude; time; azimuth and amplitude; interval to L.A.N.; latitude; longitude and time sights; identification of celestial bodies; position of single sight; Sumner line; Marc St. Hilaire; lines of position; day's work; use of H.O. 203, 208, 211, 214, Ageton (1942).

The Navy has emphasized that institutions should expand and amplify the above outlines within the scope of their facilities and the time limits imposed. Captain S. P. Fullinwider, Jr., Head of the Department of Mathematics at the United States Naval Academy, has suggested the following additional topics which might be treated. In Mathematical Analysis I and II, include the binomial theorem and the rudiments of probability; also, emphasize the solution of numerical equations by graphical and numerical methods. Where Spherical Trigonometry is introduced, include Napier's rules of circular parts and the law of cosines. In the Calculus, it would be profitable to devote two or three lessons

to Simpson's rule and its applications. In Differential Equations, considerable time could be spent upon integration in series and other approximate methods of solving differential equations.

Since the inauguration of the V-12 Program, several bulletins have been issued by the Bureau of Naval Personnel pertaining to the work in mathematics. Excerpts from these bulletins follow.

From Bulletin No. 48. "Navigation and Nautical Astronomy I and II are to be basic courses and are not designed to give any technical skill in the science of navigation. Practical navigation training is to be reserved for Naval Reserve Officer Training Corps and Midshipmen Schools.

"The purpose of Navigation I is to give the basic mathematics with emphasis on the elements of spherical trigonometry and elementary vector mathematics necessary for the student to comprehend the navigation courses he will be given later under Navy instruction. Problems may be drawn from nautical situations to keep the work alive but the emphasis shall be upon mathematical principles only. Any elementary text in Spherical Trigonometry would be an adequate guide to what is desired.

"The purpose of Navigation II is to give the student the astronomical background necessary to an understanding of the principles of celestial navigation and inherent in the practice of celestial navigation. In essence this course shall be a course in elementary nautical astronomy. Any elementary text in nautical astronomy would be an adequate guide to what is desired.

"Since neither Navigation and Nautical Astronomy I or II are courses in practical navigation, equipment as charts, navigational instruments, and other devices for the study of practical navigation will not be necessary."

From Bulletin No. 66. "It has been recognized from the beginning that there would be considerable differences in the amount and quality of mathematical preparation among students entering the V-12 Program from high school. For that reason, two separate courses in mathematics were prescribed as alternatives for the first two college terms, Mathematics I and II for those students with two or less units of high school mathematics and Mathematics III and IV for those with $2\frac{1}{2}$ or more units of high school mathematics. It was contemplated that college and university faculties would put into Mathematics I those students not qualified to attempt Mathematics III. This interpretation is hereby established as Navy policy, ... Colleges and universities are authorized to transfer V-12 freshmen enrolled in Mathematics III to Mathematics I whenever it is evident that such freshmen have inadequate preparation for Mathematics III.

"It appears that some students have entered the V-12 Program whose preparation in mathematics is inadequate even for Mathematics I. Colleges and universities are hereby authorized to establish the necessary refresher or makeup classes for such students, and to make necessary readjustments in the content and hours of subsequent classes in mathematics during the first two terms. It is required, however, that such students complete the substantial equivalent of Mathematics II by the end of their second term in order that they may be pre-

pared for the qualifying examination on the work of the first two terms which all V-12 freshmen will be required to take at the end of the second term.

"The college or university shall be sole judge as to whether or not academic credit shall be given for any refresher classes or any other special classes set up under paragraphs . . . above. Such classes, whether or not given for academic credit, may be counted in the number of hours for academic work prescribed for V-12 students.

"It is not contemplated that any V-12 student shall be permitted to remain in college more than the prescribed number of terms because of deficiency in high school preparation. If he cannot, with necessary refresher classes and other adjustments in the curriculum . . . make up his deficiencies and complete all prescribed courses by the end of his second term in college, he will be subject to separation from the program."

From Bulletin No. 67. In the case of former V-1 and V-7 students, "it is not required that any student attempt to complete in one term both halves of a mathematics or physics course which normally runs for two terms, when successful completion of the first half is a prerequisite to an understanding of the work of the second."

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE TWENTIETH ANNUAL MEETING OF THE NEBRASKA SECTION

The twentieth annual meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, Nebraska, on Saturday, May 1, 1943. Professor W. C. Brenke presided in the absence of the chairman, Professor M. A. Basoco.

There were twenty people in attendance, including the following thirteen members of the Association: A. K. Bettinger, W. C. Brenke, C. C. Camp, A. R. Congdon, H. M. Cox, W. A. Dwyer, J. M. Earl, M. G. Gaba, F. S. Harper, Ralph Hull, H. L. Rice, Lulu L. Runge, and Charles Saltzer.

The following officers were elected for the coming year: Chairman, W. A. Dwyer, Creighton University; Secretary, Lulu L. Runge, University of Nebraska; Member of Executive Committee, J. M. Earl, University of Omaha.

The following papers were presented:

1. *P-adic numbers and their arithmetics*, by Dr. Albert Neuhaus, University of Nebraska, introduced by Professor W. C. Brenke.

This paper dealt with the non-archimedian valuations of the rational numbers, the derived p-adic number fields, and the fundamental operations in these fields.

2. *Mileposts of instructional research*, by Professor H. M. Cox, University of Nebraska.

The speaker discussed the mileposts of instructional research as selected from the point of view of the classroom teacher. He advanced the opinion that the progress of instructional research represents expanding areas of consciousness of responsibility. It was also remarked that a course can be described in terms of: (1) identification of subject matter (2) measurement of student achievement; (3) evaluation of student achievement in the light of parallel or overlapping courses and in the light of all-college points of view; (4) evaluation and re-evaluation of subject-matter.

3. *The present norm of the University of Nebraska grading system*, by Professor C. C. Camp, University of Nebraska.

In this investigation Professor Camp used records of grades from 1938 to 1942, and found large fluctuations from year to year, and at different course levels. Only when he considered the percentage frequencies in regrouped data for the entire university did he find what could be called a norm. Grades in the nineties, eighties, seventies, sixties, and those below sixty exhibited percentage frequencies of 11, 41, 30, 10, and 8 respectively. The distribution showed a skewness of -0.7, and was fitted to a Pearson type *III* and a Charlier type *A* curve.

4. *The Gaussian imagery of spherical objects*, by Professor O. C. Collins, University of Nebraska, introduced by Professor W. C. Brenke.

Professor Collins discussed the screen representation of a spherical surface as projected by a system of thin lenses. He remarked that every type of perspective projection is possible by this means. A simplified construction was used for the consideration of space images of solid objects.

5. *Certain applications of Postel's projection*, by Professor O. C. Collins, University of Nebraska, introduced by Professor W. C. Brenke.

The speaker discussed the uses of a certain flat network representing the parallels and meridians of a sphere. The network was obtained by projecting the surface of the sphere azimuthally from the zenith of a point on the equator. It was pointed out that the device could be employed for the construction of a universal planisphere, as a template for the construction of world maps centered on any locality, and for the representation and solution of spherical triangles.

6. *Some remarks on the theorem of Blasius in theoretical fluid mechanics*, by Professor M. A. Basoco, University of Nebraska. (Read by title.)

The content of this paper consisted of a simple derivation of a theorem of Blasius (Zeits. f. Math. u. Phys., 1909, 10) which affords an expression for the lift and moment experienced by a cylinder placed in a moving ideal fluid. The lift and moment are expressed as contour integrals involving the complex potential describing the motion.

7. *On the flow about an elliptic cylinder*, by Charles Saltzer, University of Nebraska.

The speaker considered the application of conformal mapping to hydrodynamics in order to obtain the complex potential of the non-circulatory flow about circular and elliptic cylinders, and in order to compute the lift and moment on the cylinders as determined by Blasius' theorem

8. *Arithmetics of hypercomplex number systems*, by Professor Ralph Hull, University of British Columbia.

LULU L. RUNGE, *Secretary*

THE ELEVENTH ANNUAL MEETING OF THE WISCONSIN SECTION

The eleventh annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the University of Wisconsin in Milwaukee on Saturday, May 15, 1943. Sessions were held in the morning and in the afternoon, with the chairman of the Section, Professor R. H. Bardell, presiding. An address of welcome was delivered by Dean Frank O. Holt of the Extension Division of the University of Wisconsin. Members and their guests assembled for lunch at the Hotel Schroeder.

There were forty-eight in attendance, including the following seventeen members of the Association: R. H. Bardell, Leon Battig, Ethelwynn R. Beckwith, May M. Beenken, F. A. Butter, Jr., L. A. V. DeCleene, Fannie Hopkins, R. C. Huffer, M. L. Jautz, R. E. Langer, Mary Felice, R. E. Norris, Irene Price, G. A. Sedlak, P. L. Trump, J. I. Vass, Louise A. Wolf.

The following officers were elected for the coming year: Chairman, May M. Beenken, State Teachers College, Oshkosh; Secretary-Treasurer, P. L. Trump, University of Wisconsin; Program Committee, Leon Battig, University of Wisconsin Extension Division, Sheboygan, Louise A. Wolf, University of Wisconsin in Milwaukee. It was decided to hold the next meeting at Milwaukee-Downer College in Milwaukee in May, 1944. The choice of the exact date of the meeting was left to the host institution.

The following papers were presented:

1. *Geometrical location of the roots of certain composite equations*, by Professor Morris Marden, University of Wisconsin in Milwaukee.

Professor Marden discussed the location of the roots of the equation

$$C(z) = a_0B(0) + a_1B(1)z + \cdots + a_mB(m)z^m = 0$$

relative to the location of the roots of the equations

$$A(z) = a_0 + a_1z + \cdots + a_mz^m = 0$$

and

$$B(z) = b_0 + b_1z + \cdots + b_nz^n = 0.$$

The roots of the equation $A(z)=0$ were assumed to lie either in a ring shaped region, or in a sector bounded by two rays intersecting at an angle not greater than π radians. The roots of $B(z)=0$ were assumed to lie either between two

circles consisting of points the ratio of whose distances from the points $z=0$ and $z=m$ was constant, or between two circles consisting of points at which the segment from $z=0$ to $z=m$ subtended a constant angle. The roots of $C(z)=0$ were then found to lie in a region obtained by expanding, in a suitable manner, the region containing the roots of $A(z)=0$.

2. *The characteristics and applications of photo-cells*, by Walther Richter, American Institute of Electrical Engineers, introduced by Professor R. H. Bardell.

3. *Mathematics training for the armed forces now in progress at the University of Wisconsin*, by Professor R. E. Langer, University of Wisconsin.

4. *Mathematics in the signal corps*, by Haym Kruglak, Milwaukee Vocational School, introduced by Professor H. P. Evans.

5. *Applications of mathematics to industry*, by J. A. Deubel, Perfex Corporation, introduced by Professor Ethelwynn R. Beckwith.

P. L. TRUMP, *Secretary*

THE SECOND ANNUAL MEETING OF THE METROPOLITAN NEW YORK SECTION

The second annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Brooklyn College, Brooklyn, New York, on Saturday, May 8, 1943. Professor F. H. Miller presided at the morning session. At the afternoon session, Professor H. F. Mac Neish, Chairman of the Section, acted as general chairman, and Dr. Edna E. Kramer-Lassar, Vice-Chairman of the Section, acted as program chairman.

The attendance was about one hundred and twenty-three, including the following fifty-eight members of the Association: R. G. Archibald, L. A. Aroian, A. V. Baez, Brother Bernard Alfred (Welch), Frank Boehm, C. B. Boyer, A. B. Brown, Jewell Hughes Bushey, J. H. Bushey, Louise M. Comer, T. F. Cope, W. H. H. Cowles, Jesse Douglas, W. H. Fagerstrom, J. M. Feld, Edward Fleisher, R. M. Foster, Etta Greenberg, Harriet M. Griffin, C. C. Grove, R. A. Harrison, Solomon Hurwitz, R. A. Johnson, Sidney Kaplan, Herman Karnow, E. H. Koch, Jr., Edna E. Kramer-Lassar, Nathan Lazar, C. H. Lehmann, Herman Levy, C. C. MacDuffee, H. F. Mac Neish, P. H. Mc Grath, May Hickey Maria, A. E. Meder, Jr., Joseph Milkman, F. H. Miller, E. C. Molina, L. T. Moore, M. A. Nordgaard, Max Peters, Mina S. Rees, Moses Richardson, S. G. Roth, Arthur Sard, S. A. Schelkunoff, Edna C. Schnefel, James Singer, E. R. Stabler, J. E. Thompson, H. E. Wahlert, Israel Wallach, Alan Wayne, John Williamson, H. P. Wirth, Jack Wolfe, Margaret Y. Woodbridge, R. C. Yates.

At the beginning of the afternoon session President Harry D. Gideonse of Brooklyn College welcomed the Section to Brooklyn College. At the close of the afternoon session the following officers were elected for the coming year: Chair-

man, C. C. MacDuffee, Hunter College; Vice-Chairman, Max Peters, New Utrecht High School; Secretary, H. E. Wahlert, New York University; Treasurer, F. H. Miller, Cooper Union

The following papers were presented:

1. *Mathematics in navigation*, by Lieutenant Commander Delwyn Hyatt, USN, U. S. Merchant Marine Academy, introduced by Professor Mac Neish.

It was pointed out in this address that problems in navigation involve a considerable amount of mathematics up to and including spherical trigonometry, but that the actual practice of navigation has been reduced to the use of tables so that the navigator needs only the ability to copy numbers correctly from the tables, and to perform additions and subtractions. It was stated that the armed forces have learned that the average youth inducted in recent years has not been able to perform the simple mathematical operations required in practice.

2. *Some applications of mathematical statistics in the war effort*, by Professor Harold Hotelling, Columbia University, introduced by Dr. Kramer-Lassar.

The speaker described a number of war time applications of mathematical statistics. Among the applications mentioned were the following: determination of probabilities of hits and associated probabilities, quality control of manufactured articles by sampling inspection, cryptography, personnel placement and psychological researches, meteorology, medical and agricultural research, statistical designs in physical, chemical, and engineering research, and economic and administrative statistics. It was predicted that those who make fundamental studies of mathematical statistics with a view of its immediate usefulness will be able to utilize their knowledge after the war.

3. *Secret communications*, by Major D. D. Millikin, New York University, introduced by Mr. Wahlert.

Major Millikin gave a loosely chronological discussion of codes and ciphers, chosen for their human interest or humor, but which illustrated the devices most frequently employed for secret communications. Included in the presentation were many examples of historic interest, with special emphasis on the methods used during the period from the Revolutionary War through World War I. Applications of cryptography in the fields of literature, recreation, business, and sports were also described.

4. *Pure mathematics as a war course*, by Professor F. J. Murray, Columbia University, introduced by Professor Jewell Hughes Bushey.

The speaker called attention to the fact that our civilization is based upon techniques which are fundamentally mathematical. It was remarked that the content and sequential arrangement of the standard mathematics courses has been developed through many centuries for the purpose of solving technical problems, and in order to obtain a precise description and understanding of natural phenomena. Emphasis was placed upon the thesis that to cut or weaken the standard mathematics curriculum would be a very harmful procedure.

5. *Should the present practical trend in secondary mathematics be extended to the college mathematics curriculum?* by Dr. Nathan Lazar, Midwood High School.

6. *Should the related mathematics course be the required mathematics course for all ninth year pupils?* by Max Peters, New Utrecht High School.

In February, 1943, a special one-year course in related mathematics was introduced in the New York City high schools to give non-academic students the essential mathematics needed for success in pre-induction and post-induction training. The course included the following topics: a review of arithmetic with emphasis on applied problems; the simple properties of plane and solid geometric forms; the use of instruments such as the protractor, micrometer, vernier, and the slide rule; a unit of algebra including the equation, the formula, ratio and proportion, variation, and graphs; indirect measurement, including scale drawing, the Pythagorean theorem, and trigonometry of the right triangle; a unit on vectors. The speaker expressed the opinion that such a course with its emphasis on applications gives the student a much richer insight into the role that mathematics plays in our civilization, and permits greater flexibility in adapting the ninth year students, both academic and non-academic.

H. E. WAHLERT, *Secretary*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, New York, April 22, 1944
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO, Columbus, April 6, 1944
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, MARCH 11, 1944
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1944

INDEX TO VOLUME 50, 1943

THE AMERICAN MATHEMATICAL MONTHLY

By W. M. DAVIS, Cornell College

GENERAL MATHEMATICAL PAPERS

ALBERT, A. A. An inductive proof of Descartes' rule of signs, 178–180.

ALBERT, G. E. The closure of systems of orthogonal functions, 163–169.

BEER, F. P. A plane representation of vectors and tensors, 605–610.

BELL, E. T. Algebraic identities in the theory of numbers, 535–541.

BIRKHOFF, GARRETT. What is a lattice?, 484–487.

BLANKINSHIP, W. A. The curtain rod problem, 186–189.

BLUMENTHAL, L. M. and GILLAM, B. E. Distribution of points in n-space, 181–185.

BOYER, C. B. An early reference to division by zero, 487–491.

BUCK, R. C. Partition of space, 541–544.

CELL, J. W. Mathematics instruction in engineering, 303–307.

COOLIDGE, J. L. Three hundred years of mathematics at Harvard, 347–356.

CONKWRIGHT, N. B. An elementary proof of the Budan-Fourier theorem, 603–605.

COXETER, H. S. M. A geometrical background for de Sitter's world, 217–228.

CURRY, H. B. The Heaviside operational calculus, 365–379.

CURTISS, J. H. Convergent sequences of probability distributions, 94–105.

DEHN, MAX. Mathematics, 600 B.C.–400 B.C., 357–360.

——— Mathematics, 400 B.C.–300 B.C., 411–414.

——— and HELLINGER, E. D. Certain mathematical achievements of James Gregory, 149–163.

DRESDEN, ARNOLD. The derivatives of composite functions, 9–12, 356.

FISCHER, C. H. The rate of interest in instalment contracts, 545–547.

FISCHER, I. C. A projective construction for plane nodal cubics, 611–617.

FLEXNER, W. W. Azimuth line of position, 475–484.

FORSYTHE, ALEXANDRA. Divisors of zero in polynomial rings, 7–8.

GEIRINGER, HILDA. The geometric foundations of the mechanics of a rigid body, 492–502.

GILLAM, B. E. See Blumenthal, L. M.

GROSS, BERNHARD. On an integral transformation of general circuit theory, 90–93.

HAURWITZ, B. The applications of mathematics in meteorology, 77–84.

HEATH, R. V. A magic cube with $6n^3$ cells, 288–291.

HELLINGER, E. D. See Dehn, Max.

HINRICHSEN, J. J. L. The libration points in an n-body problem, 231–237.

JACKSON, DUNHAM. Legendre functions of the second kind and related functions, 291–302.

KELLER, M. W., SHREVE, D. R., and REMMERS, H. H. Diagnostic testing program in Purdue University, 85–90.

LANGER, R. E. What are Eigen-werte?, 279–287.

MACDUFFEE, C. C. What is a matrix?, 360–365.

MANN, H. B. Quadratic forms with linear constraints, 430–433.

MENGER, KARL. What is dimension?, 2–7.

MILLS, W. H. Iteration of the ϕ function, 547–549.

PALL, GORDON. On the product of linear forms, 173–175.

PHILLIPS, H. B. Applied mathematics, 307–309.

PRICE, G. B. Adjustments in mathematics to the impact of war, 31–34.

QUERRY, J. W. A survey course for teachers, 176–178.

RADÓ, TIBOR. What is the area of a surface?, 139–141.

REMMERS, H. H. See Keller, M. W.

RICHARDSON, R. G. D. Applied mathematics and the present crisis, 415–423.

ROOT, R. E. Mathematics and mechanics in the postgraduate school at Annapolis, 238–244.

RUNNING, T. R. Graphical solutions of cubic, quartic and quintic, 170–173.

SADOWSKY, M. A. Equiareal patterns, 35–40.

SHAPIRO, HAROLD, An arithmetic function arising from the ϕ function, 18–30.

SHREVE, D. R. See Keller, M. W.

STABLER, E. R. Boolean algebra as an introduction to postulational methods, 106–110.

THURSTON, H. S. The solution of p-adic equations, 142–148.

ULAM, S. M. What is measure?, 597–602.

WARD, L. E. Geodesics and plane arcs on an oblate spheroid, 423–429.

WHITMAN, E. A. Some historical notes on the cycloid, 309–315.

YATES, R. C. Folding the conics, 228–230.

DISCUSSIONS AND NOTES

Edited by MARIE J. WEISS, Sophie Newcomb College

ALBERT, A. A. A suggestion for a simplified trigonometry, 251–253.
AROIAN, L. A. A certain type of integral, 382–383.
BELLMAN, RICHARD. A note on the divergence of a series, 318–319.
——— A note on determinants and Hadamard's inequality, 550–551.
BIBB, S. F. An analog of Pascal's arithmetical triangle, 319–322.
BREAKWELL, J. V. See Ransom, W. R.
BROCK, J. E. An example on double series, 619.
BRUCE, J. M. Approximations to a central angle, 42–45.
CRAWFORD, W. S. H. Square roots from a table of cosines, 190–191.
GROSSMAN, H. D. The Morley triangle: a new geometric proof, 552.
HOHN, F. E. An existence proof for logarithms, 115–116.
KNEBELMAN, M. S. An elementary limit, 507.
NIVEN, IVAN. On elliptic integrals, 41–42.
RAMLER, O. J. Quadratic and cubic equations, 507–509.
RANSOM, W. R. and BREAKWELL, J. V. The shortest way found quickly, 436–438.
RECHT, L. S. The game of nim, 435.
REINER, IRVING. Functions not formulas for primes, 619–621.
ROBBINS, H. E. A note on the Riemann integral 617–618.
SCHWERDTFEGER, H. On P_r-matrices, 621.
SHAW, J. B. Quotients in Galois fields, 434.
SUTTON, R. M. An instrument for drawing confocal ellipses and hyperbolas, 253–254.
UNDERWOOD, R. S. A simple criterion for rational roots, 250–251.
WALTZ, A. K. Analogs, 380–381.
YATES, R. C. Classification of the conics, 112–115.
——— Four points, 505–506.

CLUBS AND ALLIED ACTIVITIES

Edited by J. S. FRAME, Michigan State College

TOPICS

J. S. F. Annual reports, 316.
J. S. F. Numerical integration, 244–250.
J. S. F. Solving a triangle without tables, 622.
J. S. F. Some applications of the slide rule, 55–57.

ACTIVITIES

Allegheny College, 504.
Boston University, 58.
Brooklyn College, 191–192.
Brown University, 503.
Butler University, 194.
Central Michigan College, 553.
College of Saint Teresa, 504.
Cooper Union, 193.
Duke University, 503–504.
Harvard University, 623.
Hunter College, 57–58.
Illinois State Normal University, 317.
Iowa State College, 192–193, 624.
Kansas State College, 554.
Lehigh University, 553–554.
Louisiana State University, 553.
New York University, 191.
Oberlin College, 192.
Oklahoma A. and M. College, 316–317.
Southern Illinois Normal University, 317.
St. John's University, 58.
University of Alabama, 58.
University of Buffalo, 58.
University of Chicago, 193.
University of Dayton, 316.
University of Kansas, 318, 624.
University of Missouri, 193–194.
University of Nebraska, 625.
University of North Carolina, 318.
University of Virginia, 193.
University of Wisconsin, 317, 626.
Upsala College, 626.
Wayne State Teachers College, 192.
Wayne University, 625.
Wellesley College, 554.
Winthrop College, 318.

RECENT PUBLICATIONS

Edited by VIRGIL SNYDER, Cornell University

NEW BOOKS RECEIVED

54, 119, 258–259, 325, 385, 453, 511, 559.

REVIEWS

Names of authors are in ordinary type, those of reviewers in capitals.

Agnew, R. P. *Differential Equations*. J. K. L. MACDONALD, 323–324.
Aiken, M. L. See Thompson, W. H.
Aude, H. T. R. See Evans, P. L.
Aumann, Georg. See Haupt, Otto.
Baker, Edward. *A First Course in Mathematics for Students of Engineering and the Physical Sciences*. L. S. HILL, 631–633.
Baker, H. F. *An Introduction to Plane Geometry, with Many Examples*. VIRGIL SNYDER, 441–443.
Baten, W. D. See Homes, M. C.
Baylis, C. A. See Cooley, J. C.
Beatty, Samuel, and Jenkins, J. T. *Introduction to the Calculus*, E. S. HAMMOND, 47–49.
Beisel, B. R. See Evans, P. L.
——— See Granville, W. A.
Betz, William. See Kokomoor, F. W.
Black, A. H. See Tricomi, Francesco.
Bland, J. R. See Kells, L. M.
Boothroyd, S. L. See Duncan, J. C.
Borgman, W. M. See Nelson, A. L.
Bowden, Muriel. See Hooper, A.
Boyer, L. E. See Olds, E. G.
Brillouin, L. See Jordan, E. C.
Brink, R. W. *Spherical Trigonometry*. VIRGIL SNYDER, 324–325.
Brown, O. E. See Morris, Max.
Camp, C. C. See Sherwood, G. E. F.
Cooley, H. R., Graham, P. H., John, F. W. and Tilley, A. *Logarithms, Trigonometry, Statistics*. BEATRICE L. HAGEN, 322–323.
Cooley, J. C. *A Primer of Formal Logic*. C. A. BAYLIS, 449.
Crow, E. L. See Van Leuven, E. P.
Duncan, J. C. *Essentials of Astronomy*. S. L. BOOTHROYD, 198–199.
Durfee, W. H. See Moore, G. E.
Dye, L. A. See Hartkemeier, H. P.
Eberle, W. C. See Weems, W. H.
Escott, E. B. See Fehr, H. F.
Evans, P. L. *Mathematics for Technical Training, Calculus*, H. T. R. AUDE, 254–255.
——— *Plane Trigonometry with Tables*. B. R. BEISEL, 440.
Everitt, W. L. See Jordon, E. C.
Fehr, H. F. *A Study of the Number Concept of Secondary School Mathematics*. E. B. ESCOTT, 49–51.
Flexner, W. W. and Walker, G. L. *Military and Naval Maps and Grids*. R. E. GILMAN, 195.
——— See Hart, W. L.
Folley, K. W. See Nelson. A. L.
Frame, J. S. See James, Glenn.
Gaskell, R. E. See Olds, E. G.
Gehman, H. M. See Morris, Max.
Gilman, R. E. See Flexner, W. W.
——— See Smith, Z. L.
Graham, P. H. See Cooley, H. R.
Granville, W. A., Smith, P. F. and Mikesh, J. S. *Spherical Trigonometry and Tables*. B. R. BEISEL, 438.
Gregory, Cottell. See Hart, W. W.
Gupta, Hansraj. *Symmetric Functions in the Theory of Integral Numbers*. FRITZ HERZOG, 51–52.
Hagen, Beatrice L. See Cooley, H. R.
——— See Hart, W. W.
Hall, D. W. See Knebleman, M. S.
Hammond, E. S. See Beatty, Samuel.
Hammond, J. R. *Concise Spherical Trigonometry with Applications and Reviews of Solid Geometry and Plane Trigonometry*. VIRGIL SNYDER, 555
Hart, W. L. See Hart, W. W.
——— Wilson, W. A. and Tracey, J. I. *College Mathematics*. W. W. FLEXNER, 556.
Hart, W. W. *Basic Mathematics*. M. E. MUNROE, 196–197.
——— Gregory, Cottell and Schult, Veryl. *Mathematics in Daily Use*. SARA L. NELSON, 384.
——— and Hart, W. L. *Plane Trigonometry, Solid Geometry, and Spherical Trigonometry*. BEATRICE L. HAGEN, 450–451.
Hartkemeier, H. P. *An Introduction to Managerial Business Statistics*. L. A. DYE, 443.
Hartley, M. C. See Rogers, W. W.
Haupt, Otto and Aumann, Georg. *Differential- und Integral rechnung*. SEYMOUR SHERMAN, 116–117.
Heineman, E. R. *Plane Trigonometry with Tables*. D. W. WESTERN, 383–384.
Herzog, Fritz. See Gupta, Hansraj.
Homes, M. C. *An Outline of Probability and its Uses*. W. D. BATEN, 440.
Hooper, A. *A Mathematics Refresher*. MURIEL BOWDEN, 325.
Hurwitz, H., Jr. See Lindsay, R. B.
Jackson, Dunham. See Thompson, A. J.
James, Glenn and James, R. C. *Mathematics Dictionary*. J. S. FRAME, 627–631.
James, R. C. See James, Glenn.
Jenkins, J. T. See Beatty, Samuel.
John, F. W. See Cooley, H. R.
Johnson, Marie M. See Rider, P. R.
Johnson, Roberta F. See Munshower, C. W.
Jordan, E. C., Nelson, P. H., Osterbrock, W. C., Pumphery, F. H., and Smeby, L. C. *Fundamentals of Radio*. L. BRILLOUIN, 52–54.

Keller, E. G. *Mathematics of Modern Engineering*. Vol. II. M. S. KNEBELMAN, 255–256.
Kells, L. M., Kern, W. F. and Bland, J. R. *Spherical Trigonometry with Naval and Military Applications*. G. L. WALKER, 198.
Kern, W. F. See Kells, L. M.
Knebelman, M. S. See Keller, E. G.
——— and Thomas, T. Y. *Principles of College Algebra*. D. W. HALL, 385.
Kokomoor, F. W. *Mathematics in Human Affairs*. WILLIAM BETZ, 445–446.
Lane, Ruth O. See Olds, E. G.
Lazar, Nathan. See Olds, E. G.
Lengyel, B. A. See Wolfenden, H. H.
Lerner, Samuel. See Weir, J. J.
Lindsay, R. B. *Students' Handbook of Elementary Physics*. H. HURWITZ, Jr., 557.
MacDonald, J. K. L. See Agnew, R. P.
Mikesh, J. S. See Granville, W. A.
Moore, G. E. *An Outline of College Algebra*. W. H. DURFEE, 117–118.
Morris, Max and Brown, O. E. *Differential Equations*. H. M. GEHMAN, 324.
Moye, W. B. See Sprague, A. H.
Munroe, M. E. See Hart, W. W.
Munshower, C. W. and Wardwell, J. F. *Basic College Mathematics. A General Introduction*. ROBERTA F. JOHNSON, 450.
Nelson, A. L., Folley, K. W. and Borgman, W. M. *Calculus*. MARIE J. WEISS, 195–196.
Nelson, P. H. See Jordan, E. C.
Nelson, Sara L. See Hart, W. W.
Oakley, C. O. See Phillips, H. B.
Olds, E. G., Boyer, L. E., Lane, Ruth O., Lazar, Nathan and Wren, F. L. *A Source Book of of Mathematical Applications*. R. E. GASKELL, 509–510.
Osterbrock, W. C. See Jordan, E. C.
Perkins, F. W. See Reddick, H. W.
Phillips, H. B. *Analytical Geometry and Calculus*. C. O. OAKLEY, 439.
Pumphery, F. H. See Jordan, E. C.
Reddick, H. W. *Differential Equations*. F. W. PERKINS, 447–449.
Rhodes, C. E. See Thompson, J. E.
Rider, P. R. *Plane and Spherical Trigonometry*. MARIE M. JOHNSON, 194–195.
Robson, A. *An Introduction to Analytic Geometry*. VIRGIL SNYDER, 510–511.
Rogers, W. W. and Welton, P. L. *Blueprint Reading at Work*. M. C. HARTLEY, 443.
Roudebush, Elizabeth. *Laboratory Geometry*. ELMER TOLSTED, 258.
Schult, Veryl. See Hart, W. W.
Sherman, Seymour. See Haupt, Otto.
Sherwood, G. E. F. and Taylor, A. E. *Calculus*. C. C. CAMP, 256–258.
Slocum, Frederick. See Weems, W. H.
Smeby, L. C. See Jordan, E. C.
Smith, P. F. See Granville, W. A.
Smith, Z. L. *A Review of Arithmetic*. R. E. GILMAN, 195.
Snyder, Virgil. See Baker, H. F.
——— See Brink, R. W.
——— See Hammond, J. R.
——— See Robson, A.
——— See Thompson, W. H.
——— See Vennigerholz, E.
——— See Woods, Roscoe.
Sprague, A. H. *Essentials of Plane and Spherical Trigonometry*. W. B. MOYE, 452.
Taylor, A. E. See Sherwood, G. E. F.
Thomas, T. Y. See Knebleman, M. S.
Thompson, A. J. *Table of Coefficients of Everett's Central Difference Interpolation Formula*. DUNHAM JACKSON, 451–452.
Thompson, J. E. *Elements of Spherical Trigonometry*. C. E. RHODES, 46–47.
Thompson, W. H. and Aiken, M. L. *1000 Preflight Problems*. VIRGIL SNYDER, 558.
Tilley, A. See Cooley, H. R.
Tolsted, Elmer. See Roudebush, Elizabeth.
Tracey, J. I. See Hart, W. L.
Tricomi, Francesco. *Lezieni di Analisi Matematica*. Parte Seconda. A. H. BLACK, 118.
Van Leuven, E. P. *General Trade Mathematics*. E. L. CROW, 197–198.
Vennigerholz, E. *Solution of the Trisection Problem*. VIRGIL SNYDER, 559.
Walker, G. L. See Flexner, W. W.
——— See Kells, L. M.
Wardwell, J. F. See Munshower, C. W.
Weems, W. H. and Eberle, W. C. *Learning to Navigate*. FREDERICK SLOCUM, 558.
Weir, J. J. *Elementary Mathematics for the Machine Trades*. SAMUEL LERNER, 556.
Weiss, Marie J. See Nelson, A. L.
Welton, P. L. See Rogers, W. W.
Western, D. W. See Heineman, E. R.
Wilson, W. A. See Hart, W. L.
Wolfenden, H. H. *The Fundamental Principles of Mathematical Statistics*. B. A. LENGYEL, 444–445.
Woods, Roscoe. *Analytic Geometry*. VIRGIL SNYDER, 46.
Wren, F. L. See Olds, E. G.

PROBLEMS AND SOLUTIONS

Edited by OTTO DUNKEL, Washington University, H. S. M. COXETER, University of Toronto, and ORRIN FRINK, JR., Pennsylvania State College

AUTHORS

Numbers refer to pages, black face type indicating a problem solved and solution published; italics, a problem solved, but the solution not published; ordinary type, a problem proposed.

Agnew, R. P., **71**.
Alfieri, F. A., *123*.
Allen, R. K., **202**, *203*, *391*, *455*, 513, *566*, *637*.
Anning, N. H., 130.
Aude, H. T. R., *62*, 560.
Baker, G. A., *572*.

Barjansky, Adolph, *59*, *60*, *62*.
Barrow, D. F., *202*.
Bauserman, Thomas, **336**.
Beatty, S., 264.
Becker, H. W., 200, *202*, **327**, 512.
Beckerman, Alexander, **200**.
Bellman, Richard, 65, 124, 569.
Bissinger, B. H., **61**, *200*.
Bouvaist, R., **330**.
Brock, Paul, *64*.
Browne, D. H., *59*, *120*, *123*, 261, 327, *389*, *455*, *563*, 514, *564*, *568*, *637*.
Bruce, C. W., 119, *566*.
Buck, R. C., *389*, **389**, **565**, 637.
Buker, W. E., *59*, 62, *120*, *202*, **203**, 326, *328*, *389*, *455*, *513*, *564*, *634*, *636*.
Bullard, J. A., *61*, **130**, *641*.
Butchart, J. H., 59, *262*, **387**, 456, 563, 569, 634.
Butter, F. A., Jr., **59**, *61*, *62*.
Campbell, W. B., **62**.
Carlton, Virginia, **566**.
Carver, W. B., *387*.
Cassity, C. R., *131*.
Clarke, W. B., *391*, **561**.
Clawson, J. W., **268**.
Coe, C. J., 560.
Constable, M. L., *513*, **636**.
Coşniţă, Cezar, 570, 572.
Court, N. A., 59, *60*, *70*, *121*, **125**, 200, 203, **204**, **260**, 326, *389*, 454, 456, 564, 516, *520*, 560, *571*, **576**, **636**.
Coxeter, H. S. M., **125**, 329, *641*.
Crane, R. E., *59*.
Davis, U. P., 634.
Demir, Hüseyin, 638.
Dernham, M. A., 386.
Dick, Betty, 391.
Donchian, P. S., 326.
Dorwart, H. L., **392**.
Douglas, William, *455*.
Dresden, Arnold, **639**, **640**.
Dribin, D. M., *455*.
Duarte, F. J., 560.
Dulmage, Lloyd, 512.
Dunkel, Otto, 264, 330.
Elder, J. D., *455*.
Emmons, C. W., *455*, *636*.
Erdös, P., 65, 68, 124, 330, 391.
Eriksson, Thor, *637*.
Eves, Howard, 59, *59*, **60**, *62*, *63*, *64*, **64**, *121*, *121*, **123**, *129*, **134**, 200, **202**, *203*, **260**, **261**, **262**, 386, **387**, *387*, 388, **389**, *389*, **391**, *455*, *457*, *563*, **518**, **520**, 565, *566*, **567**, *571*, **635**, *641*.
Eves, Ruth, *121*.
Feld, J. M., 125.
Field, S. E., *455*.
Fields, W. L., *455*.
Finkel, Daniel, *120*, **638**.
Fleisher, Edward, *130*.
Ford, L. R., **455**.
Foreman, W. C., *455*.
Frankel, E. T., *121*.
Frink, Orrin, Jr., *71*.
Furman, Albert, 386.
Gehman, H. M., **455**.
Glicksman, A. M., **575**.
Gould, S. H., 120, **395**, *568*.
Gove, H. E., **204**.
Greenwood, J. A., 131.
Greenwood, R. E., *455*.
Gunderson, N. G., **120**.
Halmos, P. R., 569.
Hammer, P. C., **641**.
Hamming, B., *455*.
Hastings, Cecil, Jr., *572*.
Hawthorne, Frank, 512.
Heath, R. V., 121, *203*, 328, 386, 513.
Heyda, J. F., *59*.
Hill, P. R., 202, 513.
Jamison, Free, **563**, **564**, *455*.
John, Fritz, **397**.
Kalish, Aida, *203*.
Kaplansky, Irving, **389**, *455*, **514**.
Karlin, Meyer, *455*.
Kazarinoff, D. K., 454.
Kelly, J. B., **567**.
Kelly, L. M., *60*, *62*, *64*, **121**, *123*, *200*, 260, *262*, *387*, 389, **517**, **518**, *520*, **562**.
Kraïtchik, Maurice, 122.
Langman, Harry, 124.
Larsen, H. D., **121**, *200*.
Latshaw, Elmer, *455*.
Lax, Melvin, *200*.
Linfield, B. Z., **572**.
Markham, W. H., **513**.
Matlack, David, 260.
McGavock, William, *455*, *513*.
Miller, Norman, *262*.
Mitchell, Sol, 326.
Murray, C. A., *59*.
Musselman, J. R., 64, **134**, **457**.
Näkladem, V. V., 120, **567**.
Niven, Ivan, 200, **637**.
Nordhaus, E. A., **64**, *71*.
Oakley, C. O., 454.
Peiser, A. M., **200**.
Pennell, W. O., *262*.
Penney, Walter, *455*, *568*.
Raine, P. W. A., **129**.
Robinson, Raphael, 263.
Robinson, Robin, **128**, 517.
Rosenbaum, Joseph, **63**, *123*, *200*, **201**, *387*, *391*, 512, *637*.
Rosenbaum, R. A., 200.
Rosenthal, Arthur, **568**.
Roth, S. G., *455*.
Ruderman, H. D., 516.
Rufus, W. C., *455*, 633.
Russ, Alexander, *636*.
Sadowsky, M. A., 120, 634.
Santaló, L. A., **397**.
Scheffé, Henry, **573**, *641*.
Schell, E. D., *455*.
Seidenberg, A., **68**.
Sherwood, Eugene, *634*.
Sherwood, J. E., *455*.
Singer, James, 65.
Smith, P. G., *566*.
Spencer, H. E., *455*.
Springer, C. E., **70**, *129*, 271, **563**, 564, **565**, *571*, *572*.
Starke, E. P., *59*, *60*, *61*, **62**, *62*, **63**, **66**, **71**, *120*, **122**, **328**, *389*, **390**, *455*, *513*, *563*, **564**, *566*, *568*, **634**, *636*, *637*, *641*.
Stewart, B. M., 391, 516.

Stone, A. H. *391*, 261, 454.
Strobel, C. F., **455**.
Strock, E. E., *455*.
Szmojsz, Jerzy, *563*.
Thébault, V., 59, *59*, *63*, 65, 66, 71, 120, 120, 125, *129*, 134, 200, *203*, 204, 206, *260*, 264, **264**, 268, 326, 330, 330, 333, 386, *387*, *389*, 392, 454, 457, 457, 512, 517, 518, 560, 561, 563, *567*, 569, 574, 576, 633, *636*, 638, 638.
Thomas, P. D., *60*, *62*, *123*, *200*, **202**, 260, 336, **460**, **570**, **571**, *572*.
Titus, Eugene, *566*.
Topp, C. W., *566*.
Utz, W. R., *455*.
Wall, H. S., 633.
Waltz, A. K., *130*.
Ward, Morgan, **334**, **457**.
Wayne, Alan, *391*, **455**, **513**, *566*, *636*.
Welchens, J. B., *123*.
White, M. W., *391*.
Wilensky, Michael, 259.
Williams, G. A., *455*.
Williamson, John, *71*, **271**, **131**.
Wilson, Hazel S., *566*.
Woodbridge, J. L., 59, *564*.
Yamakawa, K., *262*.
Yanosik, G. A., *64*, 121.
Yates, R. C., 123

SOLUTIONS

Numbers in black face type refer to problems; those in light face to pages.

E-512, 59–60. **E-513**, 60–61. **E-514**, 61–62. **E-515**, 62. **E-516**, 62–63. **E-517**, 63. **E-518**, 63–64. **E-519**, 64. **E-520**, 327–328. **E-521**, 64. **E-522**, 120–121. **E-523**, 121. **E-524**, 121–122. **E-525**, 122–123. **E-526**, 123–124. **E-527**, 260. **E-528**, 200–201. **E-529**, 201–202. **E-530**, 202. **E-531**, 202, 513. **E-532**, 203. **E-533**, 260–261. **E-534**, 261. **E-535**, 261–262. **E-536**, 262–263. **E-538**, 328–329. **E-539**, 388–389. **E-540**, 389. **E-542**, 389. **E-543**, 389–390. **E-544**, 390. **E-545**, 391. **E-546**, 455–456. **E-547**, 561–562. **E-548**, 513–514. **E-549**, 562. **E-550**, 514–516. **E-551**, 563. **E-552**, 563–564. **E-553**, 564. **E-554**, 564, 565. **E-555**, 565–566. **E-556**, 566–567. **E-557**, 567. **E-558**, 567–568. **E-559**, 634–635. **E-560**, 568. **E-561**, 635–636. **E-562**, 636. **E-563**, 636. **E-564**, 637.
3890, 264–267. **3970**, 268, 273. **3986**, 330–333. **4012**, 66–68. **4014**, 68–70. **4015**, 70. **4016**, 129–130. **4017**, 130–131. **4018**, 125–127. **4019**, 128–129, 517–518. **4020**, 71. **4021**, 71–72. **4023**, 131–134. **4024**, 204. **4025**, 134–136. **4026**, 333–334. **4027**, 334–335. **4028**, 460–461. **4029**, 392–395. **4031**, 395–397. **4033**, 336. **4034**, 457–460. **4036**, 397–399, **4037**, 570–571. **4038**, 518–520. **4039**, 520–521. **4040**, 571. **4041**, 572–573. **4042**, 573–574. **4044**, 574–575. **4045**, 575. **4048**, 638–639. **4049**, 576–578. **4050**, 639–640. **4051**, 640–641. **4053**, 641–643.

NEWS AND NOTICES

Edited by B. W. JONES, Cornell University

Colleges, Technical Schools, and Universities: Brown University, 341; The University of California, 341–342; The Catholic University of America, 342; The University of Chicago, 342; Columbia University, 342; The University of Illinois, 342; The State University of Iowa, 342; The University of Michigan, 342–343; The University of North Carolina, 343; The Ohio State University, 343; The University of Texas, 343; The University of Wisconsin, 343.
Mexican Congress of Mathematics, 213.
Our Retiring Secretary-Treasurer, 1.
Summer Courses, 341–343.

PERSONAL MENTION

This section contains the names of persons taking an active part in meetings, newly elected members, officers of the Association and of the various Sections, and persons mentioned in the department of News and Notices.

Adams, Louise, 578.
Ader, O. B., 461.
Adkins, J. B., 136.
Albert, A. A., 74, 461.
Alberti, Furio, 580, 592.
Alexander, C. K., 533, 578.
Alexander, Mary D., 74.
Alkire, Esther, F., 463.
Allen, Harriet W., 579.
Alvarez, I. F. J., 213.
Ames, D. B., 521.
Anderson, A. H., 592.
Anderson, E. W., 405.
Anderson, P. H., 340.
Annear, P. R., 205.
Archibald, H. F., 643.
Archibald, R. C., 74.
Archibald, R. G., 461.
Arena, J. F., 405.
Arnold, K. J., 521.
Artin, Emil, 594, 595.
Atkins, H. P., 580.
Avann, S. P., 463.
Ayre, H. G., 579.
Ayres, W. L., 340, 591.
Bacon, H. M., 337, 340, 579.
Baez, A. V., 205.
Bailey, C. W., 205.
Bailey, H. W., 136.
Baker, Frances, 591.
Baker, G. A., 579.
Ballard, Ruth Mason, 580.
Banwarth, C. S., 592.
Bardell, R. H., 652.
Barnes, W. E., 644.
Barnett, I. A., 74.
Barone, Henry, 340.
Bartram, P. R., 592.
Basoco, M. A., 651.
Battig, Leon, 652.
Beal, Juna L., 579.
Beard, Helen P., 205.
Beckenbach, E. F., 211.
Beegle, B. L., 136.
Beenken, May M., 652.
Beer, F. P., 205, 473.
Beesley, E. M., 592.
Beinert, R. L., 205.
Bennett, Theodore, 521.
Berggren, W. P., 338.
Bernstein, Dorothy L., 137.
Berry, A. C., 579.
Bert, O. F. H., 13.
Bird, M. T., 521.
Birkhoff, G. D., 591, 592.
Bissinger, B. H., 579.
Blanche, E. E., 136.
Blincoe, J. W., 579.
Bliss, G. A., 340.
Blumberg, Henry, 472.
Blumberg, J. O., 579.
Blumenthal, L. M., 74, 206.
Boardman, H. C., 405.

Boas, R. P., Jr., 521.
Borgman, W. M., Jr., 521.
Botts, T. A., 592.
Bourne, S. G., 205.
Boyd, P. P., 110.
Brader, J. L., 533.
Bradford, W. H., 579.
Bradley, A. D., 579.
Brand, Louis, 74.
Branson, Herman, 205.
Brauer, Richard, 521.
Braverman, Benjamin, 592.
Bray, H. E., 74.
Brendel, Ruth A., 463.
Brenke, W. C., 643, 650.
Bright, H. F., 399.
Brock, Paul, 137.
Brown, B. H., 592.
Brown, M. C., 110.
Brown, O. E., 472.
Browne, P. L., 580, 592.
Brunk, H. D., 592.
Buchanan, H. E., 74.
Buck, R. C., 205.
Bumer, C. T., 471.
Burkett, F. J. H., 521, 643.
Burns, H. E., 136.
Burr, I. W., 596.
Busemann, Herbert, 74, 521, 592.
Bussey, W. H., 74.
Butchart, J. H., 136.
Butter, F. A., Jr., 400.
Cabrera, Blas, 213.
Cain, W. H., 592.
Cairns, S. S., 591.
Cairns, W. D., 138, 206, 209, 273.
Calvert, R. L., 579.
Camp, C. C., 651.
Camp, E. J., 15, 16.
Caris, V. B., 471.
Carlson, C. S., 211.
Carrillo, Nabor, 213.
Carver, W. B., 74, 206, 406, 594.
Caton, W. B., 205.
Celauro, F. L., 524.
Celis, A. B., 213.
Cell, J. W., 340.
Chin, Louise H., 205.
Christe, D. E., 205.
Churchill, R. V., 74.
Civin, Paul, 592.
Clark, A. G., 531.
Clark, C. E., 643.
Clarke, F. Marian, 212.
Clippinger, R. F., 13, 529.
Coble, A. B., 74.
Coburn, Nathaniel, 522.
Coe, C. J., 74, 406, 408, 596.
Cohen, A. C., Jr., 405.
Cohen, L. W., 74.
Cole, Nancy, 273.
Coleman, J. B., 212, 643.
Coleman, W. B., 273.
Collins, O. C., 651.
Conklin, E. G., 136.
Conkwright, N. B., 74, 206.
Cooley, H. R., 461.
Copeland, A. H., 522, 594.
Corwin, Margaret T., 590.
Costello, G. R., 463.
Court, N. A., 74.
Cox, H. M., 651.
Coxeter, H. S. M., 74, 206, 522.
Craig, H. V., 136.
Cramer, Paul, 341.
Crane, Rufus, 471, 472.
Crawford, I. C., 591.
Crawford, S. C., 13.
Crawford, W. S. H., 16.
Crow, E. L., 205.
Curry, H. B., 14, 210.
Curtiss, D. R., 461.
Curtiss, J. H., 273.
Dancer, Wayne, 406.
D'Arco, Paul, 592.
Daus, P. H., 534.
Davis, E. A., 530.
Davis, H. T., 399.
Davis, J. E., 210.
Davis, Margaret R., 405.
Davis, W. M., 74, 206.
DeCicco, John, 522.
Dehn, Max, 579.
DeMoss, M. L., 592.
Deubel, J. A., 653.
Dinkines, Flora, 212, 592.
Dittman, F. W., 274.
Dobbie, J. W., 340.
Doob, J. L., 74.
Dorwart, H. L., 13, 15, 530.
Douglas, Jesse, 399, 405.
Downing, H. H., 110.
Doyle, W. C., 474, 592.
Dresden, Arnold, 593.
Dressler, B. B., 405.
Duncan, D. C., 533.
Dunford, Nelson, 461.
Dunkel, Otto, 74, 206.
Dunlap, P. R., 592.
Durand, Janet C., 524.
Durfee, W. H., 463.
Durham, R. L., 592.
Dwyer, W. A., 650.
Earl, J. M., 650.
Eaves, J. C., 111.
Edward Joseph, Sister, 405.
Eisenhart, L. P., 136.
Elkins, T. A., 13.
Epstein, Benjamin, 205.
Erdös, Paul, 461.
Erickson, R. W., 137.
Esposito, J. P., 592.
Evans, G. C., 138.
Evans, H. B., 137.
Evans, H. P., 137.
Eves, H. W., 205.
Fair, L. A., 110.
Farnell, A. B., 592.
Ferguson, W. E., 473.
Fernández, C. F., 213.
Ficken, F. A., 137.
Fine, N. J., 137.
Finkel, B. F., 206.
Flexner, W. W., 591.
Ford, L. R., 206, 340, 595.
Fox, A. H., 643.
Frame, J. S., 206, 461.
Francis, S. A., 405.
Franklin, Philip, 74.
Fraser, W. A., 405.
Freese, Frances, 463.
Frink, Orrin, Jr., 74, 206.
Fulton, D. G., 522.
Gage, W. H., 522.
Gail, Albert, 15.
Gándara, A. N., 213.
Garfin, Louis, 644.
Garrison, L. M., 643.
Gass, C. B., 592.
Geiringer, Hilda, 210.
Gelbart, Abe, 461.
Gentry, F. C., 340, 461.
Gerst, Irving, 592.
Gibson, R. W., 643.
Gideonse, H. D., 653.
Gillette, E. F., 137.
Gingrich, C. H., 15.
Givens, J. W., Jr., 205.
Goldstine, H. H., 273.
Golomb, Michael, 592.
Gouwens, Cornelius, 15.
Graustein, Mary C., 273.
Graves, L. M., 138.
Green, Agnes A., 522.
Green, J. W., 462.
Grove, V. G., 73.
Groves, Marjorie J., 206.
Haag, V. H., 205.
Hailperin, Theodore, 592.
Hall, D. W., 273.
Halmos, P. R., 461.
Hamilton, Lola R., 463
Hammer, Sigmund, 529.
Hamming, R. W., 205.
Hanna, J. R., 405.
Hardman, W. R., 462.
Harshbarger, Frances, 137.
Hart, W. L., 462.
Harvey, G. G., 522.
Hassler, J. O., 531, 532.
Hastings, Cecil, Jr., 205.
Hatfield, Charles, 110.
Hatfield, Charles, Jr., 405.
Haurwitz, B., 592.
Hausmann, B. A., 592.
Hawthorne, Frank, 592.
Hay, G. E., 405.
Heath, R. V., 74.
Hedlund, G. A., 591.
Hedrick, E. R., 73.
Helly, Edward, 400, 592.
Herzog, Fritz, 579.
Hestenes, M. R., 73, 74, 206.
Heyda, J. F., 340.
Hicks, H. C., 529.
Hildebrand, F. B., 522.
Hildebrandt, E. H. C., 643.
Hildner, R. C., 399.
Hills, E. J., 533, 593.
Hoel, P. G., 533.
Holt, E. W., 211.
Holt, F. O., 652.
Holt, Marshall, 13.
Horn, W. J., 405.
Hotelling, Harold, 654.
Householder, A. S., 522.
Hove, E. Marie, 524.
Hovey, B. K., 206.
Howard, Mrs. A. S., 110.
Howard, Tryphena, 340.
Howell, S. W., 206.
Howie, J. M., 340.
Huff, G. B., 522.
Hull, Ralph, 579.
Hunter, Louise S., 206.
Hutchings, Phyllis H., 579.
Hutchings, W. L., 462.
Hyatt, Delwyn, 654.
Irick, Paul, 463.
Jackson, Dunham, 15, 74.
Jacobson, Nathan, 579.
James, R. D., 522.
Jamison, Free, 405.
Jenkins, E. D., 211.
Jennings, S. A., 580.
Jensen, C. M., 644.
Johanson, R. N., 273.
John, Fritz, 110, 273.
Johnson, Marie M., 522.
Johnson, R. A., 74.
Johnston, Ernest, 463.
Jonah, D. A., 274.
Jones, B. W., 205, 206.
Jones, P. S., 400, 407.
Jordan, H. J., 593.
Justice, H. K., 472.
Justin, E. M., 274.
Kac, Mark, 522.
Kalisch, G. K., 593.
Kamarad, Bessie V., 463.
Kaplan, Sidney, 212.
Kaplansky, Irving, 206.
Karpinski, L. C., 74.
Karst, O. J., 593.
Keller, M. W., 594, 596.
Kempner, A. J., 74, 530, 531.
Kendall, Claribel, 522.
Kerr, F. J., 593.
Kiefer, E. C., 596.
Kiefer, Lois, 644.
Kieval, H. S., 644.
Killen, C. G., 73.
Kingston, J. M., 644.
Kiokemeister, Fred, 644.
Kirchen, C. J., 137, 462.
Kliphardt, R. A., 593.
Koken, J. C., 593.
Kopp, P. J., 593.
Krabill, D. M., 136.
Kramer-Lassar, Edna E., 653.
Krathwohl, W. C., 407.
Kruglak, Haym, 653.
Kuhn, H. W., 462.
Kyle, G. D., 15.

Lafferty, W. A., 462.
LaFon, J. E., 74.
Langer, R. E., 74, 206, 653.
Latimer, C. G., 111.
Lazar, Nathan, 655.
Lee, Mary Ann, 462.
Lehner, Joseph, 273.
Lengyel, B. A., 462.
LeVeque, W. J., 405, 531.
Levy, Sophia H., 337, 338.
Lewis, A. J., 532.
Lipsich, H. D., 593.
Lubin, C. I., 462.
Luginbuhl, R. T., 405, 644.
Lund, Jeanette, 463.
Lundberg, G. H., 405.
Lunn, A. C., 73.
MacDuffee, C. C., 206, 522, 591, 654.
Mackey, G. W., 580.
Mac Lane, Saunders, 206.
Mac Neish, H. F., 653.
Maddaus, Ingo, Jr., 206.
Maddrill, J. D., 593.
Male, C. T., 643.
Mallory, A. E., 530.
Mandel, John, 405.
Mandelbrojt, Szolem, 593.
Mann, H. B., 644.
Manning, Rhoda, 522.
Marden, Morris, 652.
Maria, Deborah M., 274.
Marks, Helen, 529.
Marth, Ella, 405.
Martin, W. T., 405, 462.
Matchett, Margaret S., 531.
McBrien, V. O., 206, 274.
McCarty, A. L., 338, 406.
McCoy, N. H., 74.
McDonald, J. H., 338.
McEwen, G. F., 534.
McFadden, Leonard, 206
McLaughlin, K. F., 274.
McShane, E. J., 74, 591.
Menger, Karl, 74, 595.
Meyer, H. L., 400.
Mickle, E. J., 593.
Middlemiss, R. R., 473.
Miekle, P. T., 644.
Milgram, A. N., 340.
Miller, F. H., 644, 653, 654.
Miller, Pearl C., 580
Millikin, D. D., 654.
Mills, W. H., 405.
Milne, W. E., 74.
Miser, H. J., 580, 593.
Mode, E. B., 73.
Molina, E. C., 274.
Montgomery, Deane, 522.
Moore, M. G., 644.
Moore, W. L., 110.
Moorman, R. H., 212.
Moots, E. E., 73.
Morkovin, Vladimir, 644.
Morris, F. R., 338.
Morris, Richard, 590, 591.
Morse, Marston, 138.
Moulton, E. J., 74, 206, 399.
Moursund, A. F., 462.
Munro, W. D., 274.
Munroe, M. E., 254.
Murnaghan, F. D., 591.
Murray, F. J., 654.
Musselman, J. R., 74, 206.
Myers, W. H., 399.
Nápoles, Alfonso, 405, 521.
Neisius, W. V., 593.
Nelson, C. A., 523.
Nelson, N. D., 399.
Nelson, Sara L., 137.
Nemenyi, Paul, 530.
Neuhaus, Albert, 650.
Nevins, W. V., III, 405.
Newell, H. E., 523.
Newsom, C. V., 74, 206.
Newton, Abba V., 523.
Niersbach, P. M., 533.
Niven, Ivan, 74, 462.
Nordhaus, E. A., 399.
Norman, P. B., 137, 463.
Northrop, E. P., 523.
Novak, J. D., 644.
Nowlan, Mabel I., 341.
Oakley, C. O., 210.
O'Connor, R. E., 137.
Ogawa, George, 593.
Oldenburger, Rufus, 74.
Olds, E. G., 14.
Olmsted, J. M. H., 274.
Opatowski, Isaac, 16.
Owens, Helen B., 400.
Pall, Gordon, 523.
Palmer, H. A., 593.
Paxson, E. W., 137, 407.
Peebles, N. A., 206.
Pell, W. H., 644.
Pelletier, Arthur, 461.
Pepper, P. M., 523, 594, 596.
Perlin, I. E., 74.
Peterhans, W. A., 593.
Peters, Max, 405, 654, 655.
Peters, Ruth M., 462.
Pflaum, C. W., 593.
Phillips, H. B., 74.
Piranian, George, 524, 593.
Pollard, Harry, 523.
Pollard, W. G., 206.
Polya, George, 338, 405.
Poppen, H. A., 405.
Porter, Ruth E., 274.
Price, G. B., 523, 591.
Priester, G. C., 15.
Puckett, W. T., 137.
Putnam, A. L., 593.
Quinn, Grace S., 274.
Radó, Tibor, 74, 471.
Rainich, G. Y., 407.
Raney, G. N., 580.
Rauch, L. L., 524.
Rauch, S. E., 462.
Raynor, G. E., 210.
Recht, A. W., 530.
Reckzeh, J. K., 212.
Reid, W. P., 463.
Reid, W. T., 73, 74, 595.
Reingold, Haim, 73, 580, 593.
Rex, E. C., 212.
Rice, H. L., 405.
Richardson, R. G. D., 590.
Rickart, C. E., 463.
Richter, Walther, 653.
Rider, P. R., 213.
Ridings, Marie, 463.
Riess, J. K., 523, 593.
Rietz, H. L., 212.
Riordan, John, 593.
Roberts, G. G., 110.
Rock, D. H., 644.
Rodabaugh, L. D., 523.
Roessler, E. B., 336, 337.
Rommel, J. D., Jr., 110.
Rosenbloom, P. C., 274, 462.
Rosenthal, Arthur, 212.
Rosser, J. B., 212.
Roth, S. G., 644.
Rowland, Annie N., 206.
Rowland, J. J., 463.
Runge, Lulu L., 650, 652.
Running, T. R., 406.
Rupp, C. A., 341.
Rusk, W. J., 137.
Russell, B. M., 206.
Russell, Helen G., 137.
Russi, F. Z., 213.
Sadowsky, M. A., 74, 340, 580.
Salem, Ralph, 523.
Salem, Raphael, 593.
Salkind, Charles, 593.
Saltzer, Charles, 405, 651.
Samelson, Hans, 461.
Sarton, G. A., 73.
Schatten, Robert, 644.
Schilling, O. F. G., 74, 523.
Schneckenburger, Edith R., 523.
Schoenberg, I. J., 210, 591.
Schultz, O. T., 405.
Schweigert, G. E., 644.
Scibiorski, A. C., 340.
Scobert, W. G., 206, 463.
Scott, E. J., 400, 405.
Seebeck, C. L., 137, 580.
Shanks, M. E., 474.
Shephard, R. W., 463.
Sheridan, L. W., 274.
Silber, Jack, 405.
Singer, Maurice, 206.
Sleight, E. R., 407.
Smiley, M. F., 73.
Smith, A. J., 274, 523.
Smith, C. M., 400.
Smith, C. V. L., 399.
Smith, F. C., 462.
Smith, J. C., 593.
Smith, R. E., 400.
Snyder, Virgil, 206.
Snyder, W. S., 74, 580.
Sobczyk, Andrew, 212.
Sollins, A. D., 593.
South, D. E., 112.
Spooner, C. C., 523.
Stabler, E. R., 406.
Steen, F. H., 274, 462.
Steenrod, Norman, 74.
Steinhaus, H. W., 206.
Stewart, B. M., 407, 523.
Stewart, E. C., 206.
Stewart, I. D., 523.
Stokes, Ellen C., 212.
Stone, M. H., 74, 137, 138.
Sturm, R. G., 12, 14, 529.
Sumner, Ruth G., 337.
Sutton, R. M., 16.
Synge, J. L., 462.
Szegö, Gabor, 337.
Tamarkin, J. D., 592.
Tarski, Alfred, 73.
Taub, A. H., 523.
Taylor, C. F., 405.
Taylor, J. S., 14.
Thielman, H. P., 15, 16, 73.
Thomas, Evan, 340.
Thorne, C. J., 644.
Throckmorten, V. C., 212, 406.
Thullen, Peter, 593.
Toney, H. S., 406.
Treiber, H. I., 206.
Trigg, C. W., 463.
Trump, P. L., 652, 653.
Turner, Lois K., 206.
Ullsvik, B. R., 400.
Underhill, A. L., 15, 17.
Upton, C. B., 137.
Uspensky, J. V., 338.
Utz, W. R., Jr., 206.
Vallarta, M. S., 213.
Valuckas, Carol C., 593.
Vedora, G. C., 463.
Vest, M. L., 524.
Wade, T. L., Jr., 580.
Wahlert, H. E., 654, 655.
Waider, K. J., 338.
Walker, L. A., 406.
Walker, R. J., 74, 212.
Wall, H. S., 74.
Wallace, A. D., 462.
Wallach, Israel, 593.
Wallick, E. E., 593.
Walter, P. M., 13.
Ward, Morgan, 532, 533.
Weaver, J. H., 74.
Weaver, Warren, 138.
Wegner, K. W., 15.
Wehausen, J. V., 473.
Weinstein, Alexander, 524.
Weiss, Marie J., 74, 206.
Wells, N. W., 463.
Wells, V. H., 400.
Wetzel, Marion D., 644.
Wheeler, C. R., 400.
Whiteman, A. L., 595.
Whitman, P. M., 210, 211.
Whyburn, G. T., 138.
Whyburn, W. M., 533.
Widder, D. V., 137.

Wiegand, J. T., 406.
Wilcox, L. R., 74, 580.
Wilder, R. L., 74.
Wiley, F. B., 471.
Wiley, J. W., 206.
Wilkins, J. E., 400.
Williams, W. L., 212.
Wilson, W. E., 400.
Wolf, Frantisek, 15.
Wolf, Louise A., 652.
Wolfe, R. S., 406.
Wood, F. M., 524, 593.
Wood, Fredrick, 336.
Woods, Roscoe, 212.
Wylie, C. R., Jr., 471.
Wyman, Max, 463.
Young, P. M., 400.
Youngs, J. W. T., 591.
Zant, J. H., 340.
Zippin, Lee, 463.
Zuckerman, H. S., 524.
Zygmund, Antoni, 591.

NECROLOGY

Bolza, Oskar, 137.
Campbell, A. D., 74.
Currier, C. H., 212.
DeLong, I. M., 74.
Dodd, E. L., 212.
Fubini, Guido, 463.
Granville, W. A., 274.
Hackney, Lilian, 341.
Harter, G. A., 463.
Haskins, C. N., 74.
Hawkes, H. E., 463.
Hedrick, E. R., 138, 409–411.
Helly, Edward, 644.
Hilbert, David, 341.
Locke, L. L., 644.
McGiffert, James, 524.
Menuet, R. L., 463.
Metzler, W. H., 463.
Minssen, H. F., 341.
Mitchell, H. H., 341.
Neikirk, L. L., 400.
Osgood, W. F., 463.
Passano, L. M., 274.
Reilly, J. F., 74.
Risley, N. S., 274.
Scarborough, J. H., 212.
Showman, H. M., 463.
Smith, Clara E., 463.
Tyler, M. H., 341.
Van Vleck, E. B., 580.
Westemeier, J. J., 341.
White, H. S., 463.
Williams, J. E., 463.

WAR INFORMATION

Edited by C. V. NEWSOM, University of New Mexico.

A bibliography of cryptography, 345–346.
A new deferment procedure for mathematicians, 586–588.
Bibliography on artillery, 404.
Committee on available teachers of collegiate mathematics, 528.
Engineering, science, and management war training, 276–278.
Essential mathematics for minimum army needs, 580, 585.
From selective service, 214, 275, 527.
From the war department, 214.
From the war policy committee, 466.
Important bills before congress, 525–527.
National committee on physicists and mathematicians, 400–403.
Pre-induction courses in mathematics, 465.
Retraining programs for teachers, 275–276.
Service opportunities for women, 524–525.
Suggestions from the office of scientific personnel of the national research council, 75–76.
The army specialized training program, 466–470.
The Chicago institute of military studies, 403–404.
The Curtiss-Wright engineering cadette program, 276.
The Michigan division for emergency training, 464–465.
The Minnesota program in pre-flight mathematics, 344–345.
The Navy college training program, 645–650.
The problem of securing teachers of collegiate mathematics for wartime needs, 215–216.
The United States armed forces institute, 464.
The war policy committee, 138.
Women in aerology and electronics, 470.

REPORTS OF THE MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

MEETINGS OF THE ASSOCIATION

Election to membership, W. D. Cairns, 205–206; W. B. Carver, 405–506; 592–593.
Board of governors, W. D. Cairns, 205–209.
The twenty-sixth summer meeting, W. B. Carver, 588–594.

MEETINGS OF ITS SECTIONS

Edited by N. B. CONKWRIGHT, University of Iowa

Allegheny Mountain Section, April 1942 meeting, H. L. DORWART, 12–15; April 1943 meeting, H. L. DORWART, 529–530.
Illinois, Michigan, and Indiana Sections, April 1943 meeting, E. C. KIEFER, C. J. COE, and M. W. KELLER, 594–596.
Kentucky Section, April 1942 meeting, D. E. SOUTH, 110–112.
Metropolitan New York Section, May 1943 meeting, H. E. WAHLERT, 653–655.
Michigan Section, March 1942 meeting, C. J. COE, 406–408.
Minnesota Section, May 1942 meeting, A. L. UNDERHILL, 15–17.
Missouri Section, December 1942 meeting, M. E. SHANKS, 473–474.
Nebraska Section, May 1943 meeting, LULU L. RUNGE, 650–652.
Northern California Section, January 1943 meeting, H. M. BACON, 336–340.
Ohio Section, April 1943 meeting, RUFUS CRANE, 471–472.
Philadelphia Section, November 1942 meeting, P. M. WHITMAN, 210–211.
Rocky Mountain Section, April 1943 meeting, A. J. LEWIS, 530–532.
Southern California Section, March 1943 meeting, P. H. DAUS, 532–534.
Wisconsin Section, May 1943 meeting, P. L. TRUMP, 652–653.

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INDEX TO VOLUME 51, 1944

THE AMERICAN MATHEMATICAL MONTHLY

By W. M. Davis, Cornell College

GENERAL MATHEMATICAL PAPERS

Ayers, W. L. Interesting the engineering student, 200–205.
Beiler, A. H. An electrical Chinese ring puzzle, 133–137.
Bradshaw, J. W. More modified series, 389–391.
Brink, R. W. College mathematics during reconstruction, 61–74.
Bruck, R. H. See Wade, T. L.
Cairns, S. S. Deformations of plane rectilinear complexes, 247–252.
Campbell, J. W. Motion with respect to moving axes, 377–381.
Cole, R. H. Associated frequency distributions in biometry, 252–261.
Dehn, Max. Mathematics, 300 B.C.–200 B.C., 25–31.
——— Mathematics, 200 B.C.–600 A.D., 149–157.
Derry, Douglas. Affine geometry of convex quartics, 78–83.
Evans, H. P. Mathematics textbooks for the introductory college courses, 12–18.
Eves, Howard. Concerning some perspective triangles, 324–331.
Frame, J. S. Tangent triangles to a biquadratic curve, 445–450.
Halmos, P. R. The foundations of probability, 493–510.
Higgins, T. J. Biographies and collected works of mathematicians, 433–445.
Jackson, Dunham. The harmonic boundary value problem for an ellipse or an ellipsoid, 555–563.
Kosambi, D. D. The geometric method in mathematical statistics, 382–389.
Mahler, K. See Segre, B.
McBrien, V. O. Cardioids associated with a cyclic quadrangle, 74–77.
Min, Szu-Hoa. Non-analytic functions, 510–517.
Mordell, L. J. Rational points on cubic curves and surfaces, 332–339.
Ore, Oystein. Mathematics for students of the humanities, 453–458.
Ott, E. R. Difference equations in average value problems, 570–578.
Parker, W. V. and Pryor, J. E. Polygons of greatest area inscribed in an ellipse, 205–209.
Pryor, J. E. See Parker, W. V.
Rajagapol, C. T. The Abel-Dini and allied theorems, 566–570.
Riordan, John. Three-line Latin rectangles, 450–452.
Schwerdtfeger, H. Skew-symmetric matrices and projective geometry, 137–148.
Segre, B. and Mahler, K. On the densest packing of circles, 261–270.
Stabler, E. R. Boolean representation theory, 129–132.
Synge, J. L. Focal properties of optical and electromagnetic systems, 185–200.
Taylor, A. E. Differentiation of Fourier series and integrals, 19–25.
Wade, T. L. and Bruck, R. H. Types of symmetries, 123–129.
Wilcox, L. R. An application of a theorem of Sylvester, 270–273.
Wilder, R. L. The nature of mathematical proof, 309–323.
Wilson, E. B. Note on the t-test, 563–566.
Wylie, C. R., Jr. Hilbert's axioms of plane order, 371–376.
Youngs, J. W. T. Curves and surfaces, 1–11.

DISCUSSION AND NOTES

Edited by Marie J. Weiss, Sophie Newcomb College

Adams, O. S. Notes on the geometry of the triangle, 85–87.
Bellman, Richard. A note on the product of linear forms, 161–162.
Borman, J. S. A remark on integration by parts, 32–33.
Brand, Louis. The eight-point circle and the nine-point circle, 84–85.
Brenner, Joel. Determination of latitude in an emergency, 343–344.
Craig, H. V. A vectorial development of two differentiation formulas, 347.
——— An acknowledgement, 581.
Curtis, H. B. A derivation of Cardan's formula, 35.
Dennis, F. L. A slide rule solution of oblique spherical triangles, 159–161.
Dubisch, Roy. The roots of a Cayley number, 278.
Durham, R. L. A simple construction for the